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SATELLITE ORBITS ABOUT A PLANET WITH ROTATIONAL SYMMETRY

BY

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ABSTRACT

A closed-form solution is obtained for the orbit of an Earth satellite which is influenced by the third harmonic in the potential in addition to the two dominant oblateness terms. The results are carried to the "second order" in the rates of the secular variations and in the periodic fluctuations. The secular equations are solved in closed form in two cases: when the inclination is close to 63.4° and when it is not. In the former case, the stability and period of the slow oscillation of the perigee are discussed.

1. INTRODUCTION

The objective of this paper is to derive an approximate closed-form solution, of high accuracy and without restriction on the inclination angle or eccentricity, to the equations of motion of an Earth satellite.

It has been shown by Roberson (1)[2] that the motion of a near satellite around an oblate Earth may be rather simply described to first order in the oblateness by introducing a slowly rotating frame of reference, as follows:

(i) A "mean" orbital plane having constant inclination, rotates slowly with constant angular velocity about the polar axis.

(ii) The angular frame of reference within this plane also rotates slowly with constant angular velocity.

With respect to this doubly moving frame, the motion of the satellite differs from pure Keplerian elliptic motion only by perturbations which are periodic, including a small periodic motion out of the mean orbital plane.

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

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That such a description is possible is, indeed, suggested by more classical perturbation procedures, for example, Krause (2), which show that the only "secular" motions due to first-order oblateness are the precession of the nodes and the rotation of the line of apsides (at rates with which Roberson's treatment agrees).

A description somewhat similar to Roberson's has been developed by King-Hele (3), without the small periodic motion out of the reference plane. King-Hele's formulation of the motion of the node, however, is unsatisfactory for non-zero "eccentricity."

It is the purpose of this paper to obtain a description of the orbit beyond first order. Classical procedures indicate that one must expect, in general, slow variations in angular momentum, eccentricity, and orbital inclination as well as in the line of nodes and the line of apsides. Therefore, we examine certain differential equations, derivable from the dynamical situation, satisfied by the orbital perturbations as described relative to a more general, slowly moving reference frame. As in Roberson's treatment, the slow motion of the reference frame is almost completely determined by the requirement that the orbital perturbations, the solutions of the differential equations, shall be "periodic," that is, all the secular orbit changes are absorbed into the slowly moving reference frame. However, in dealing with the interaction between the Earth's "pearness" and its oblateness, it is found convenient to choose the inclination of the reference plane in such a way that the "orbit plane" undergoes a slow oscillation about the reference plane as the perigee moves around.

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We will assume that the Earth has rotational symmetry and that the motion of a satellite is governed solely by the Earth's potential function; that is, other forces like drag and attraction of the sun and moon are not considered. We introduce a right-handed inertial rectangular coordinate system with the origin at the center of mass of the Earth and the z-axis coincident with the rotational axis and positive in the north direction. The positive x-axis is chosen in the direction of the vernal equinox (of, say, 1950). The potential $V(r, \delta)$ of a point in space is a function of the distance r to the center of mass and declination δ. The function V is given by

$$V = \frac{\mu}{r}\left[1 + \frac{J_2 R_e^2}{3r^2}(1 - 3s_\delta^2) + \frac{J_3 R_e^3}{2r^3} s_\delta (3 - 5s_\delta^2) + \frac{J_4 R_e^4}{35r^4}(3 - 30s_\delta^2 + 35s_\delta^4) + \cdots\right] \quad (1)$$

where R_e is the equatorial radius of the Earth and μ, J_2, J_3, J_4 are constants. We abbreviate $\sin\delta$ by s_δ. The odd harmonic involving J_3 is due to the possible lack of symmetry of the Earth with respect to the equatorial plane.

In deriving a closed-form solution, the accuracy and complexity of the results and the amount of effort required depend in a large measure on the choice of dependent variables and the choice of an auxiliary independent variable. These variables, in turn, should determine the position of the satellite in terms of some commonly accepted and readily understood parameters such as right ascension λ, declination δ, and radial distance r to the center of the Earth. The end result then should be judged on the complexity and accuracy with which λ, δ, and r are determined.

On the other hand, the constants which appear in a closed form solution do not have to be simply related to initial conditions if the purpose of the solution is for tracking and prediction of a satellite. In this case they are simply constants to be determined by observations using some statistical technique.

II. EQUATIONS OF MOTION

The unit vectors along the positive x-, y-, and z-axes are denoted by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively. A moving right-handed orthogonal set of unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is introduced by

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = Q \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} \tag{2}$$

where Q is the Euler matrix

$$Q = \begin{bmatrix} C_\omega C_\Omega - S_\omega S_\Omega C_\gamma & C_\omega S_\Omega + S_\omega C_\Omega C_\gamma & S_\omega S_\gamma \\ -S_\omega C_\Omega - C_\omega S_\Omega C_\gamma & -S_\omega S_\Omega + C_\omega C_\Omega C_\gamma & C_\omega S_\gamma \\ S_\gamma S_\Omega & -S_\gamma C_\Omega & C_\gamma \end{bmatrix} \tag{3}$$

with Euler angles Ω, γ, and ω.

It may be helpful for the reader to visualize Ω, γ, ω as prescribed functions of time such that Ω is the right ascension of the ascending node, γ the inclination angle, and ω the argument of perigee. However, these functions are chosen in a very special way, and their exact determination is not obtained until the end of this paper. Actually, they represent these (vaguely defined) physical parameters only in some sort of average sense. A second moving set of right-handed orthogonal unit vectors $\mathbf{e}_1^*$, $\mathbf{e}_2^*$, $\mathbf{e}_3^*$ is introduced by

$$\begin{bmatrix} \mathbf{e}_1^* \\ \mathbf{e}_2^* \\ \mathbf{e}_3^* \end{bmatrix} = Q^* \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \tag{4}$$

where

$$Q^* = \begin{bmatrix} c_\phi c_\theta & c_\phi s_\theta & s_\phi \\ -s_\theta & c_\theta & 0 \\ -s_\phi c_\theta & -s_\phi s_\theta & c_\phi \end{bmatrix} \tag{5}$$

in the transformation matrix to spherical coordinates relative to the set $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

Again it may be helpful for the reader to visualize θ as the true anomaly and ϕ as the angular deviation, as seen from the origin, of the satellite from the slowly moving plane determined by $\mathbf{e}_1$ and $\mathbf{e}_2$.

We will choose θ as the auxiliary independent variable and, by primes, denote derivatives with respect to θ, for example, $\omega' = \dfrac{d\omega}{d\theta}$. A dot over a dependent variable will denote its derivative with respect to time, for example, $\dot{r} = \dfrac{dr}{dt}$.

The variables θ and ϕ are determined such that

$$\mathbf{r} = r\mathbf{e}_1^* \tag{6}$$

where $\mathbf{r}$ is the radius vector of the satellite and

$$\ddot{r} = A\mathbf{e}_1^* + B\mathbf{e}_2^* + C\mathbf{e}_3^*. \tag{7}$$

Since $\mathbf{e}_i^*$ are orthogonal unit vectors,

$$\begin{bmatrix} \dot{\mathbf{e}}_1^* \\ \dot{\mathbf{e}}_2^* \\ \dot{\mathbf{e}}_3^* \end{bmatrix} = \begin{bmatrix} 0 & \alpha^* & \beta^* \\ -\alpha^* & 0 & \delta^* \\ -\beta^* & -\delta^* & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1^* \\ \mathbf{e}_2^* \\ \mathbf{e}_3^* \end{bmatrix} \tag{8}$$

where

$$\begin{bmatrix} 0 & \alpha^* & \beta^* \\ -\alpha^* & 0 & \delta^* \\ -\beta^* & -\delta^* & 0 \end{bmatrix} = \dot{Q}^* Q^{*-1} + Q^* \dot{Q} Q^{-1} Q^{*-1}. \tag{9}$$

From Eqs. 7 and 8 we have

$$\left.\begin{aligned} \ddot{r} - r(\alpha^{*2} + \beta^{*2}) &= A, \\ 2\dot{r}\alpha^* + r\dot{\alpha}^* - \beta^*\delta^* r &= B, \\ 2\dot{r}\beta^* + r\dot{\beta}^* + \alpha^*\delta^* r &= C. \end{aligned}\right\} \tag{10}$$

The three dependent variables which will be chosen are ρ, σ, and τ given by

$$\rho = \frac{1}{rc_\phi} \tag{11}$$

$$\sigma = \tan \phi \tag{12}$$

$$\tau = \dot{\theta}\rho^{-2}. \tag{13}$$

One can then compute

$$\left.\begin{aligned} \alpha^* &= \tau\rho^2(1+\sigma^2)^{-1/2}(1+F_3+F_1\sigma) \\ \beta^* &= \tau\rho^2[\sigma'(1+\sigma^2)^{-1}+F_2] \\ \delta^* &= \tau\rho^2(1+\sigma^2)^{-1/2}[(1+F_3)\sigma - F_1] \end{aligned}\right\} \tag{14}$$

where

$$\left.\begin{aligned} F_1 &= -\gamma' c_{\theta+\omega} - s_\gamma \Omega' s_{\theta+\omega}, \\ F_2 &= \gamma' s_{\theta+\omega} - s_\gamma \Omega' c_{\theta+\omega}, \\ F_3 &= \omega' + c_\gamma \Omega'. \end{aligned}\right\} \tag{15}$$

The differential equations (10) may now be written in terms of the dependent variables ρ, σ and τ.

$$\rho'' + \rho(1+F_3)^2 + \frac{\rho'\tau'}{\tau} + \rho\left[F_2^2 + 2F_2\sigma' + F_1F_3\sigma + \sigma(F_1 + F_2') + \frac{F_2\tau'\sigma}{\tau}\right] = \frac{(\sigma C - A)(1+\sigma^2)^{-1/2}}{\tau^2\rho^2}, \tag{16}$$

$$\sigma'' + \sigma(1+F_3)^2 + (F_2' - F_1 - F_1F_3) + \frac{\tau'}{\tau}(\sigma' + F_2) + 2F_2\sigma\sigma' - F_1^2\sigma + \sigma^2(F_2' + F_1) + \frac{\sigma^2F_2\tau'}{\tau} + F_1F_3\sigma^2 = \frac{C(1+\sigma^2)^{1/2}}{\tau^2\rho^2} \tag{17}$$

$$(1+F_3)\tau' + F_1(\sigma\tau)' + \tau(F_3' + F_1F_2) + \tau F_1\sigma' + \tau\sigma(F_1' - F_2 - F_2F_3) = \frac{B}{\rho^3\tau}. \tag{18}$$

The above are the general equations of motion in our coordinate

system but we will consider only the case corresponding to the Earth's potential, Eq. 1, where

$$\left.\begin{aligned} A &= \frac{\partial V(r,\delta)}{\partial r}, \\ B &= \frac{S_\gamma C_{\theta+\omega}}{rC_\delta}\frac{\partial V(r,\delta)}{\partial \delta} \\ C &= \frac{(1+\sigma^2)^{-1/2}(C_\gamma - \sigma S_\gamma S_{\theta+\omega})}{rC_\delta}\frac{\partial V(r,\delta)}{\partial \delta} \end{aligned}\right\} \tag{19}$$

The radial distance r, right ascension λ, and declination δ of the satellite are given by

$$\left.\begin{aligned} r &= \rho^{-1}(1+\sigma^2)^{1/2} \\ \lambda &= \Omega + \tan^{-1}\left[\frac{S_{\theta+\omega}C_\gamma - \sigma S_\gamma}{C_{\theta+\omega}}\right], \\ S_\delta &= (1+\sigma^2)^{-1/2}(\sigma C_\gamma + S_\gamma S_{\theta+\omega}). \end{aligned}\right\} \tag{20}$$

Up to this point no approximations have been made and the differential equations and other expressions are exact. In the next section Eqs. 16, 17, and 18 will be solved approximately when A, B, and C are given by Eq. 19.

III. A CLOSED-FORM SOLUTION

If the potential V is given simply by $V = \mu/r$ and Ω, γ, and ω are chosen to be constant, then Eqs. 16, 17, and 18 are given by

$$\left.\begin{aligned} \rho'' + \rho &= \frac{\mu}{\tau^2}(1+\sigma^2)^{-3/2}, \\ \sigma'' + \sigma &= 0, \\ \tau' &= 0. \end{aligned}\right\} \tag{21}$$

We may then choose as a closed-form solution to Eqs. 16, 17, and 18

$$\left.\begin{aligned} \rho &= \frac{\mu}{h^2}(1+\epsilon C_\theta), \\ \sigma &= 0, \\ \tau &= h, \end{aligned}\right\} \tag{22}$$

where $\Omega' = \gamma' = \omega' = \epsilon' = h' = 0$ and by Eq. 20

$$\left.\begin{aligned} r &= \rho^{-1} \\ \lambda &= \Omega + \tan^{-1}\left[\frac{C_\gamma S_{\theta+\omega}}{C_{\theta+\omega}}\right], \\ S_\delta &= S_\gamma S_{\theta+\omega}. \end{aligned}\right\} \tag{23}$$

The particular constants Ω, γ, ω, ϵ, h which enable any solution to be written in this form are then to be determined by observation of the satellite.

Returning to the equations with V given by Eq. 1, the problem is to determine Ω, γ, ω, ϵ, h as slowly varying functions of θ such that any solution to Eqs. 16, 17, and 18 has the form

$$\left.\begin{aligned} \rho &= \frac{\mu}{h^2}(1 + \epsilon C_\theta + a_1 C_\theta + \bar{a}_1 S_\theta + \rho_1) \\ \rho_1 &= a_0 + \sum_{i\geq 2} a_i C_{i\theta} + \sum_{i\geq 2} \bar{a}_i S_{i\theta} \end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned} \sigma &= b_1 S_{\theta+\omega} + \bar{b}_1 C_{\theta+\omega} + \sigma_1 \\ \sigma_1 &= b_0 + \sum_{i\geq 2} b_i S_{i\theta+\omega} + \sum_{i\geq 2} \bar{b}_i C_{i\theta+\omega} \end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned} \tau &= h(1 + F_3)^{-1}(1 + \tau_1) \\ \tau_1 &= \sum_{i\geq 1} d_i C_{i\theta+2\omega} + \sum_{i\geq 1} \bar{d}_i S_{i\theta+2\omega} \end{aligned}\right\} \tag{26}$$

where a_i, $\bar{a}_i$, b_i, $\bar{b}_i$, d_i, $\bar{d}_i$ in the expressions for ρ_1, σ_1 and τ_1 are expressed in ascending powers of the J_i's with coefficients which are functions of Ω, γ, ω, ϵ, and h. The four terms a_1, $\bar{a}_1$, b_1 and $\bar{b}_1$ play a very special role which we will discuss presently.

We adopt the following convention as to which terms of the solution are to be retained. Terms with J_2 as a factor will be considered as first order, those with J_2^2, J_3, or J_4 as a factor will be considered second order. Terms higher than the second order will not be retained in this paper, consequently V in Eq. 1 will be restricted to those terms explicitly written. For conciseness, we will write

$$J_i = J_i\left(\frac{R_e\mu}{h^2}\right)^i. \tag{27}$$

It is clear that Ω', γ', ω', ϵ', h' and a_i, $\bar{a}_i$, b_i, $\bar{b}_i$, d_i, $\bar{d}_i$ are of the first order, at least. For the moment, let us consider terms of, at most, the

first order. The equations then become

$$\left.\begin{aligned} \rho'' + \rho(1 + 2F_3) + \frac{\rho'\tau'}{\tau} &= -\frac{A}{\tau^2\rho^3}, \\ \sigma'' + \sigma + F_2' - F_1 &= \frac{C}{\tau^2\rho^3}, \\ \tau' &= \frac{B}{\tau\rho^3}. \end{aligned}\right\} \tag{28}$$

Replacing ρ, σ, τ by Eqs. 24, 25 and 26 and equating coefficients, we find that Ω', ω', d_1, d_2, d_3, b_0, b_2, a_0, a_2, $\bar{a}_2$, a_3, $\bar{a}_3$, a_4, $\bar{a}_4$ are the only terms of the first order and that among the derivatives of these terms, only b_0', a_0', a_2', $\bar{a}_2'$, a_3', $\bar{a}_3'$, a_4', $\bar{a}_4'$ are of the second order, the others being at least the third order.

Proceeding now to a second-order analysis, the equations become

$$\rho'' + \rho(1 + F_3)^2 + \frac{\rho'\tau'}{\tau} + \rho[F_2^2 + 2F_2\sigma' + \sigma(F_1 + F_2')] = \frac{(\sigma C - A)}{\tau^2\rho^2}(1 + \sigma^2)^{-1/2}, \tag{29}$$

$$\sigma'' + \sigma(1 + 2F_3) + (F_2' - F_1 - F_1F_3) + \frac{\tau'}{\tau}(\sigma' + F_2) = \frac{C}{\tau^2\rho^3}, \tag{30}$$

$$(1 + F_3)\tau' + F_1(\sigma\tau)' + \tau F_1F_2 + \tau F_1\sigma' + \tau\sigma(F_1' - F_2) = \frac{B}{\rho^3\tau}. \tag{31}$$

It is now necessary to discuss the basic philosophy which motivates our approach to this problem. For the moment, suppose a_1, $\bar{a}_1$, b_1, $\bar{b}_1$ are set equal to zero in Eqs. 24 and 25. Our procedure then would be to choose the rates Ω', γ', ω', ϵ', h' such that they eliminate the resonance (first harmonic terms in θ) in Eqs. 16 and 17 and the constant (slowly varying) term in Eq. 18. In this manner we obtain five "secular" equations which may strongly interact with each other for critical values of the variables (for example, γ near 63.4°), but the functions ρ_1, σ_1, τ_1 will in all cases have the same form. The problem is then completely transferred to the solution of these secular equations. However, a basic difficulty arises from this procedure which, as far as we have carried it, is due entirely to the presence of the odd harmonic term involving J_3 in the potential function.

When we gather the coefficient of $s_{\theta+\omega}$ in Eq. 17, the dominant term in Ω' is $2s_\gamma\Omega'$, but also there will appear terms involving J_3 which do not have s_γ as a factor. Similarly, in Eq. 16 the coefficient of c_θ will

have the dominant term $2\epsilon(\omega' + c_\gamma\Omega')$ and also certain J_3 terms which do not have ϵ as a factor. This procedure then leads to secular equations which strongly interact for small eccentricities or small inclination angles. Moreover, the unfortunate feature is that in the calculation of the coefficients in ρ_1, σ_1, and τ_1 as explicit functions of ϵ, ω, γ, and h there will appear terms which have either ϵ or s_γ in the denominator. To the second order, this does not occur, but these would appear if third-order terms had been retained. However, these higher order terms should also be retained when either ϵ or γ is small. Consequently, the treatment is unsatisfactory in these two cases.

This difficulty is removed by introducing the terms $\bar{a}_1 s_\theta$ in Eq. 24 and $\bar{b}_1 c_{\theta+\omega}$ in Eq. 25. The derivative $\bar{b}_1'$ will appear in the coefficient of $s_{\theta+\omega}$ in Eq. 17. Its purpose is to remove all terms which do not have s_γ as a factor. Similarly, the derivative $\bar{a}_1'$ will appear in the coefficient of c_θ in Eq. 16, and its purpose is to remove all terms which do not have ϵ as a factor.

The additional introduction of $a_1 c_\theta$ in Eq. 24 and $b_1 s_{\theta+\omega}$ in Eq. 25 is not an essential feature, but it is convenient to do so. For, if we did not do this, we would find that the amplitude of the fluctuation in ϵ or γ does not become small when the corresponding variable is small, again due to the presence of the J_3 terms. This means, for instance, that for small eccentricity it would, at times, be necessary to interpret ϵ as being negative. Therefore, a_1 and b_1 are used to keep the rate ϵ' proportional to ϵ and γ' proportional to s_γ.

Summarizing the philosophy of our procedure, we require that a_1, $\bar{a}_1$, b_1, $\bar{b}_1$ be functions of γ, ω, ϵ, and h such that every remaining coefficient of a first harmonic term in θ in Eq. 16 has ϵ as a factor, and every remaining coefficient of a first harmonic term in Eq. 17 has s_γ as a factor. The rates ω', Ω', ϵ', γ', and h' are then chosen such that they remove all remaining first harmonic terms in θ in Eqs. 16 and 17 and also the constant (slowly varying) terms in Eq. 18.

Now the choice of a_1, $\bar{a}_1$, b_1, $\bar{b}_1$ is not necessarily unique, for it is possible for instance, to absorb into these variables some additional terms in Eq. 17 having s_γ as a factor. The choice that was made is

$$\left.\begin{aligned} b_1 &= \frac{3}{4}\frac{J_3}{J_2} c_\gamma \epsilon s_\omega, \\ \bar{b}_1 &= \frac{3}{4}\frac{J_3}{J_2} c_\gamma \epsilon c_\omega, \\ a_1 &= -\frac{3}{4}\frac{J_3}{J_2} s_\gamma (1-\epsilon^2) s_\omega, \\ \bar{a}_1 &= -\frac{3}{4}\frac{J_3}{J_2} s_\gamma (1-\epsilon^2) c_\omega, \end{aligned}\right\} \tag{32}$$

and consequently

$$\left.\begin{aligned} b_1 s_{\theta+\omega} + \bar{b}_1 c_{\theta+\omega} &= \frac{3}{4}\frac{J_3}{J_2} c_\gamma \epsilon c_\theta, \\ a_1 c_\theta + \bar{a}_1 s_\theta &= -\frac{3}{4}\frac{J_3}{J_2} s_\gamma (1-\epsilon^2) s_{\theta+\omega} \end{aligned}\right\} \tag{33}$$

Returning now to the second-order equations, we first solve Eq. 31, then Eq. 30 and finally Eq. 29. In the computation, b_1, $\bar{b}_1$, a_1, $\bar{a}_1$ are treated as first-order terms. Consequently, the reader will find them involved in the computation of d_1, $\bar{d}_1$, d_2, $\bar{d}_2$, d_3, $\bar{d}_3$, b_0, b_2, $\bar{b}_2$, a_0, a_2, $\bar{a}_2$, a_3, $\bar{a}_3$, a_4, $\bar{a}_4$ in the J_3 terms. In the following pages of our results we have evaluated the above coefficients explicitly according to our choice in Eq. 32.

The "secular equations" are given by:

$$\begin{aligned} \Omega' = &- J_2 c_\gamma - \frac{J_2^2 c_\gamma}{2}\left(3 - \frac{13}{3}s^2_\gamma\right) \\ &- \frac{J_2^2 c_\gamma \epsilon^2}{6}\left(1 + \frac{5}{4}s^2_\gamma\right) - \frac{J_2^2 c_\gamma \epsilon^2}{4}\left(\frac{7}{3} - 5s^2_\gamma\right) c_{2\omega} \\ &- \frac{3}{2} J_4 c_\gamma \left(1 + \frac{3}{2}\epsilon^2\right)\left(\frac{4}{7} - s^2_\gamma\right) + \frac{3}{2} J_4 c_\gamma \epsilon^2 \left(\frac{3}{7} - s^2_\gamma\right) c_{2\omega} \\ &- 3 J_3 s_\gamma c_\gamma \epsilon s_\omega. \end{aligned} \tag{34}$$

$$\begin{aligned} \omega' = &\; J_2\left(2 - \frac{5}{2}s^2_\gamma\right) \\ &+ \frac{J_2^2}{2}\left[8 - \frac{133}{6}s^2_\gamma + \frac{335}{24}s^4_\gamma + \frac{\epsilon^2}{6}\left(7 - \frac{9}{2}s^2_\gamma - \frac{45}{8}s^4_\gamma\right)\right] \\ &+ \frac{J_2^2 c_{2\omega}}{4}\left[-\frac{s^2_\gamma}{3} + \epsilon^2\left(\frac{7}{3} - \frac{79}{6}s^2_\gamma + \frac{45}{4}s^4_\gamma\right)\right] \\ &+ 3J_4\left[\frac{4}{7} - \frac{31}{14}s^2_\gamma + \frac{7}{4}s^4_\gamma + \frac{9}{2}\epsilon^2\left(\frac{1}{7} - \frac{1}{2}s^2_\gamma + \frac{3}{8}s^4_\gamma\right)\right] \\ &+ \frac{3}{2} J_4 c_{2\omega}\left[\frac{3}{7}s^2_\gamma - \frac{1}{2}s^4_\gamma - \epsilon^2\left(\frac{3}{7} - \frac{5}{2}s^2_\gamma + \frac{9}{4}s^4_\gamma\right)\right] \\ &+ \frac{15}{2} J_3 s_\gamma \epsilon\left(1 - \frac{5}{4}s^2_\gamma\right) s_\omega \end{aligned} \tag{35}$$

$$\gamma' = \frac{J_2^2 s_\gamma c_\gamma \epsilon^2}{4}\left(\frac{7}{3} - \frac{5}{2}s^2_\gamma\right) s_{2\omega} - \frac{3}{4} J_4 s_\gamma c_\gamma \epsilon^2 \left(\frac{6}{7} - s^2_\gamma\right) s_{2\omega} \tag{36}$$

$$\epsilon' = -\frac{J_2^2}{4} s^2_\gamma \epsilon s_{2\omega} \left[\frac{1}{3} - \epsilon^2\left(\frac{7}{3} - \frac{5}{2} s^2_\gamma\right)\right] + \frac{3}{4} J_4 s^2_\gamma \epsilon (1 - \epsilon^2)\left(\frac{6}{7} - s^2_\gamma\right) s_{2\omega} \quad (37)$$

$$\frac{h'}{h} = \frac{J_2^2 s^2_\gamma \epsilon^2}{4}\left(\frac{7}{3} - \frac{5}{2} s^2_\gamma\right) s_{2\omega} - \frac{3}{4} J_4 \epsilon^2 s^2_\gamma \left(\frac{6}{7} - s^2_\gamma\right) s_{2\omega} + \frac{3}{2} J_3 s_\gamma \epsilon \left(1 - \frac{5}{4} s^2_\gamma\right) c_\omega. \quad (38)$$

The coefficients d_i, $\bar{d}_i$ in Eq. 26 are given by:

$$d_1 = \frac{J_2 s^2_\gamma \epsilon}{2} - \frac{J_2^2 s^2_\gamma \epsilon}{2}\left(5 - \frac{11}{2} s^2_\gamma\right) + \frac{2}{3} J_2^2 s^2_\gamma \epsilon \left(1 - \frac{35}{32} s^2_\gamma\right) c_{2\omega} + \frac{3}{2} J_4 s^2_\gamma \epsilon \left(1 + \frac{\epsilon^2}{4}\right)\left(\frac{6}{7} - s^2_\gamma\right) + \frac{J_4}{16} s^4_\gamma \epsilon^3 c_{2\omega} - \frac{J_4}{8} s^2_\gamma \epsilon^3 \left(\frac{6}{7} - s^2_\gamma\right) c_{4\omega} - \frac{3}{2} J_3 s_\gamma \left[\left(1 - s^2_\gamma - \frac{11}{16}\epsilon^2 s^2_\gamma\right) s_\omega - \frac{\epsilon^2 s^2_\gamma}{16} s_{3\omega}\right] \quad (39)$$

$$\bar{d}_1 = \frac{2}{3} J_2^2 s^2_\gamma \epsilon \left(1 - \frac{35}{32} s^2_\gamma\right) s_{2\omega} - \frac{J_4}{16} s^4_\gamma \epsilon^3 s_{2\omega} - \frac{J_4}{8} s^2_\gamma \epsilon^3 \left(\frac{6}{7} - s^2_\gamma\right) s_{4\omega} + \frac{3}{2} J_3 s_\gamma \left[\left(1 - s^2_\gamma - \frac{\epsilon^2 s^2_\gamma}{16}\right) c_\omega - \frac{\epsilon^2 s^2_\gamma}{16} c_{3\omega}\right] \quad (40)$$

$$d_2 = \frac{J_2 s^2_\gamma}{2} + \frac{J_2^2 s^2_\gamma}{2}\left(\frac{1}{2} - s^2_\gamma\right) + \frac{J_2^2 s^2_\gamma \epsilon^2}{12}\left(1 - \frac{5}{2} s^2_\gamma\right) + \frac{J_2^2 s^2_\gamma \epsilon^2}{3}\left(1 - \frac{35}{32} s^2_\gamma\right) c_{2\omega} + \frac{J_4}{2} s^2_\gamma \left(1 + \frac{3}{2}\epsilon^2\right)\left(\frac{6}{7} - s^2_\gamma\right) + \frac{3}{16} J_4 s^4_\gamma \epsilon^2 c_{2\omega} + \frac{9}{8} J_3 s^3_\gamma \epsilon s_\omega \quad (41)$$

$$\bar{d}_2 = \frac{J_2^2 s^2_\gamma \epsilon^2}{3}\left(1 - \frac{29}{32} s^2_\gamma\right) s_{2\omega} - \frac{3}{16} J_4 s^4_\gamma \epsilon^2 s_{2\omega} + \frac{3}{4} J_3 s^3_\gamma \epsilon c_\omega \quad (42)$$

$$d_3 = \frac{J_2 S^2_\gamma \epsilon}{6} + \frac{J_2^2 S^2_\gamma \epsilon}{18}\left(5 - \frac{11}{2} S^2_\gamma\right) - \frac{J_2^2}{8} S^4_\gamma \epsilon C_{2\omega}$$
$$+ \frac{J_4}{2} S^2_\gamma \epsilon \left(1 + \frac{\epsilon^2}{4}\right)\left(\frac{6}{7} - S^2_\gamma\right) + \frac{J_4}{4} S^4_\gamma \epsilon \left(1 + \frac{\epsilon^2}{4}\right) C_{2\omega}$$
$$+ \frac{J_3 S^3_\gamma}{8}\left(4 + \frac{15}{4}\epsilon^2\right) S_\omega \quad (43)$$

$$d_3 = \frac{J_2^2}{8} S^4_\gamma \epsilon S_{2\omega} - \frac{J_4}{4} S^4_\gamma \epsilon \left(1 + \frac{\epsilon^2}{4}\right) S_{2\omega} + \frac{J_3 S^3_\gamma}{8}\left(4 + \frac{13}{4}\epsilon^2\right) C_\omega \quad (44)$$

$$d_4 = \frac{J_2^2 S^2_\gamma \epsilon^2}{48}\left(1 - \frac{S^2_\gamma}{2}\right) - \frac{J_2^2 S^4_\gamma}{12}\left(1 + \frac{7}{8}\epsilon^2\right) C_{2\omega}$$
$$+ \frac{3}{16} J_4 S^2_\gamma \epsilon^2 \left(\frac{6}{7} - S^2_\gamma\right) + \frac{J_4}{8} S^4_\gamma \left(1 + \frac{3}{2}\epsilon^2\right) C_{2\omega} + \frac{15}{32} J_3 S^3_\gamma \epsilon S_\omega \quad (45)$$

$$d_4 = \frac{J_2^2 S^4_\gamma}{12}\left(1 + \frac{7}{8}\epsilon^2\right) S_{2\omega} - \frac{J_4}{8} S^4_\gamma \left(1 + \frac{3}{2}\epsilon^2\right) S_{2\omega} + \frac{15}{32} J_3 S^3_\gamma \epsilon C_\omega \quad (46)$$

$$d_5 = -\frac{J_2^2}{16} S^4_\gamma \epsilon C_{2\omega} + \frac{J_4}{40} \epsilon^3 S^2_\gamma \left(\frac{6}{7} - S^2_\gamma\right)$$
$$+ \frac{3}{20} J_4 S^4_\gamma \epsilon \left(1 + \frac{\epsilon^2}{4}\right) C_{2\omega} + \frac{3}{32} J_3 S^3_\gamma \epsilon^2 S_\omega \quad (47)$$

$$d_5 = \frac{J_2^2}{16} S^4_\gamma \epsilon S_{2\omega} - \frac{3}{20} J_4 S^4_\gamma \epsilon \left(1 + \frac{\epsilon^2}{4}\right) S_{2\omega} + \frac{3}{32} J_3 S^3_\gamma \epsilon^2 C_\omega \quad (48)$$

$$d_6 = -\frac{\epsilon^2 S^4_\gamma}{48}\left(\frac{J_2^2}{2} - 3J_4\right) C_{2\omega} \quad (49)$$

$$d_6 = \frac{\epsilon^2 S^4_\gamma}{48}\left(\frac{J_2^2}{2} - 3J_4\right) S_{2\omega} \quad (50)$$

$$d_7 = \frac{J_4}{112} \epsilon^3 S^4_\gamma C_{2\omega} \quad (51)$$

$$d_7 = -\frac{J_4}{112} \epsilon^3 S^4_\gamma S_{2\omega}. \quad (52)$$

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The coefficients b_i, $\bar{b}_i$ in Eq. 25 are given by:

$$b_0 = -J_2 S_\gamma C_\gamma \epsilon S_\omega + J_2^2 S_\gamma C_\gamma \epsilon S_\omega \left(2 - \frac{15}{4} S^2_\gamma\right) - \frac{9}{2} J_4 S_\gamma C_\gamma \epsilon \left(1 + \frac{\epsilon^2}{4}\right)\left(\frac{4}{7} - S^2_\gamma\right) S_\omega - \frac{J_4}{8} S^3_\gamma C_\gamma \epsilon^3 S_{3\omega} + \frac{3}{2} J_3 C_\gamma \left[1 - 2S^2_\gamma - \epsilon^2 S^2_\gamma + \frac{3}{8} \epsilon^2 S^2_\gamma C_{2\omega}\right] \quad (53)$$

$$b_2 = \frac{J_2 S_\gamma C_\gamma \epsilon}{3} + J_2^2 S_\gamma C_\gamma \epsilon \left(\frac{2}{9} - \frac{1}{8} S^2_\gamma\right) - \frac{3}{4} J_2^2 S^3_\gamma C_\gamma \epsilon C_{2\omega} + \frac{3}{2} J_4 S_\gamma C_\gamma \epsilon \left(1 + \frac{\epsilon^2}{4}\right)\left(\frac{4}{7} - S^2_\gamma\right) + \frac{J_4}{2} S_\gamma C_\gamma \epsilon \left(S^2_\gamma - \frac{\epsilon^2}{7} + \frac{\epsilon^2 S^2_\gamma}{2}\right) C_{2\omega} + J_3 C_\gamma S^2_\gamma (1 + \epsilon^2) S_\omega \quad (54)$$

$$\bar{b}_2 = -\frac{3}{4} J_2^2 S^3_\gamma C_\gamma \epsilon S_{2\omega} + \frac{J_4}{2} S_\gamma C_\gamma \epsilon \left(S^2_\gamma + \frac{\epsilon^2}{7}\right) S_{2\omega} - J_3 C_\gamma S^2_\gamma \left(1 + \frac{\epsilon^2}{2}\right) C_\omega \quad (55)$$

$$b_3 = \frac{J_2^2}{48} S_\gamma C_\gamma \epsilon^2 \left(1 + \frac{5}{4} S^2_\gamma\right) - \frac{5}{24} J_2^2 S^3_\gamma C_\gamma \left(1 + \frac{3}{4} \epsilon^2\right) C_{2\omega} + \frac{9}{32} J_4 S_\gamma C_\gamma \epsilon^2 \left(\frac{4}{7} - S^2_\gamma\right) + \frac{J_4}{8} S^3_\gamma C_\gamma \left(1 + \frac{3}{2} \epsilon^2\right) C_{2\omega} + \frac{15}{32} J_3 S^2_\gamma C_\gamma \epsilon S_\omega \quad (56)$$

$$\bar{b}_3 = -\frac{5}{24} J_2^2 S^3_\gamma C_\gamma \left(1 + \frac{3}{4} \epsilon^2\right) S_{2\omega} + \frac{J_4}{8} S^3_\gamma C_\gamma \left(1 + \frac{3}{2} \epsilon^2\right) S_{2\omega} - \frac{15}{32} J_3 S^2_\gamma C_\gamma \epsilon C_\omega \quad (57)$$

$$b_4 = -\frac{7}{72} J_2^2 S^3_\gamma C_\gamma \epsilon C_{2\omega} + \frac{J_4}{40} S_\gamma C_\gamma \epsilon^3 \left(\frac{4}{7} - S^2_\gamma\right) + \frac{J_4}{10} S^3_\gamma C_\gamma \epsilon \left(1 + \frac{\epsilon^2}{4}\right) C_{2\omega} + \frac{J_3}{16} S^2_\gamma C_\gamma \epsilon^2 S_\omega \quad (58)$$

$$b_4 = -\frac{7}{72} J_2^2 S^3_\gamma C_\gamma \epsilon S_{2\omega} + \frac{J_4}{10} S^3_\gamma C_\gamma \epsilon \left(1 + \frac{\epsilon^2}{4}\right) S_{2\omega} - \frac{J_3}{16} S^2_\gamma C_\gamma \epsilon^2 C_\omega \quad (59)$$

$$b_5 = -\frac{S^3_\gamma C_\gamma \epsilon^2}{32}\left(\frac{7}{18} J_2^2 - J_4\right) C_{2\omega} \quad (60)$$

$$\bar{b}_5 = -\frac{S^3_\gamma C_\gamma \epsilon^2}{32}\left(\frac{7}{18} J_2^2 - J_4\right) S_{2\omega} \quad (61)$$

$$b_6 = \frac{J_4}{280} S^3_\gamma C_\gamma \epsilon^3 C_{2\omega} \quad (62)$$

$$\bar{b}_6 = \frac{J_4}{280} S^3_\gamma C_\gamma \epsilon^3 S_{2\omega}. \quad (63)$$

The coefficients a_i, $\bar{a}_i$ in Eq. 24 are given by:

$$\begin{aligned} a_0 = {} & J_2\left(1 - \frac{3}{2} S^2_\gamma\right)\left(1 + \frac{\epsilon^2}{2}\right) + \frac{J_2}{8} S^2_\gamma \epsilon^2 C_{2\omega} \\ & + J_2^2\left[2 - \frac{13}{2} S^2_\gamma + \frac{109}{24} S^4_\gamma + \epsilon^2\left(1 - \frac{11}{6} S^2_\gamma + \frac{11}{96} S^4_\gamma\right)\right] \\ & - J_2^2 S^2_\gamma \epsilon^2\left(\frac{43}{12} - 4 S^2_\gamma\right) C_{2\omega} \\ & + \frac{3}{7} J_4\left(1 - 5 S^2_\gamma + \frac{35}{8} S^4_\gamma\right)\left(1 + 3\epsilon^2 + \frac{3}{8}\epsilon^4\right) \\ & + 3 J_4 S^2_\gamma \epsilon^2\left(\frac{6}{7} - S^2_\gamma\right)\left(1 + \frac{\epsilon^2}{6}\right) C_{2\omega} + \frac{J_4}{128} S^4_\gamma \epsilon^4 C_{4\omega} \\ & + \frac{3}{4} J_3 S_\gamma \epsilon\left[5 - 7 S^2_\gamma + \epsilon^2\left(1 - \frac{39}{16} S^2_\gamma\right)\right] S_\omega \\ & + \frac{5}{64} J_3 S^2_\gamma \epsilon^2 S_{3\omega} - \frac{27}{64}\left(\frac{J_3}{J_2}\right)^2 C^2_\gamma \epsilon^2 \end{aligned} \quad (64)$$

$$\begin{aligned} a_2 = {} & -\frac{J_2}{6}\epsilon^2\left(1 - \frac{3}{2} S^2_\gamma\right) - \frac{J_2}{6}\left(1 + \frac{3}{2}\epsilon^2\right) S^2_\gamma C_{2\omega} \\ & - \frac{J_2^2 \epsilon^2}{6}\left[2 - S^2_\gamma - \frac{41}{16} S^4_\gamma\right] \\ & + \frac{J_2^2 S^2_\gamma}{9}\left[2 - S^2_\gamma + \frac{37}{8}\epsilon^2\left(5 - \frac{11}{2} S^2_\gamma\right)\right] C_{2\omega} - \frac{J_2^2}{48} S^4_\gamma \epsilon^2 C_{4\omega} \end{aligned}$$

Continued on next page

$$- \frac{J_4}{7}\epsilon^2\left(3+\frac{\epsilon^2}{2}\right)\left(1-5s^2{}_\gamma+\frac{35}{8}s^4{}_\gamma\right)$$
$$- J_4 s^2{}_\gamma\left(\frac{6}{7}-s^2{}_\gamma\right)\left(\frac{1}{2}+2\epsilon^2+\frac{11}{32}\epsilon^4\right)c_{2\omega}$$
$$- \frac{J_4}{16}s^4{}_\gamma\epsilon^2\left(1+\frac{\epsilon^2}{2}\right)c_{4\omega}$$
$$- \frac{J_3}{4}s_\gamma\epsilon\left[3\left(3-\frac{7}{2}s^2{}_\gamma\right)+\epsilon^2\left(1-\frac{13}{4}s^2{}_\gamma\right)\right]s_\omega$$
$$- \frac{J_3}{8}s^3{}_\gamma\epsilon\left(1+\frac{27}{8}\epsilon^2\right)s_{3\omega}+\frac{9}{64}\left(\frac{J_3}{J_2}\right)^2 c^2{}_\gamma\epsilon^2 \quad (65)$$

$$\bar{a}_2 = \frac{J_2}{6}s^2{}_\gamma\left(1+\frac{3}{2}\epsilon^2\right)s_{2\omega}$$
$$- \frac{J_2^2 s^2{}_\gamma}{9}\left[2-s^2{}_\gamma+\frac{37}{8}\epsilon^2\left(5-\frac{11}{2}s^2{}_\gamma\right)\right]s_{2\omega}$$
$$+ \frac{J_2^2}{48}s^4{}_\gamma\epsilon^2 s_{4\omega}+J_4 s^2{}_\gamma\left(\frac{6}{7}-s^2{}_\gamma\right)\left(\frac{1}{2}+2\epsilon^2+\frac{9}{32}\epsilon^4\right)s_{2\omega}$$
$$+ \frac{J_4}{16}s^4{}_\gamma\epsilon^2\left(1+\frac{\epsilon^2}{2}\right)s_{4\omega}$$
$$- \frac{J_3}{4}s_\gamma\epsilon\left[3\left(3-\frac{7}{2}s^2{}_\gamma\right)+\epsilon^2\left(1-\frac{23}{8}s^2{}_\gamma\right)\right]c_\omega$$
$$- \frac{J_3}{8}s^3{}_\gamma\epsilon\left(1+\frac{27}{8}\epsilon^2\right)c_{3\omega} \quad (66)$$

$$a_3 = -\frac{5}{24}J_3 s^2{}_\gamma\epsilon c_{2\omega}+\frac{J_2^2\epsilon^2}{16}\left(\frac{1}{3}-3s^2{}_\gamma+\frac{25}{8}s^4{}_\gamma\right)$$
$$+ \frac{J_2^2 s^2{}_\gamma\epsilon}{24}\left[\frac{1}{3}\left(14-\frac{43}{4}s^2{}_\gamma\right)+\frac{\epsilon^2}{8}\left(11-\frac{19}{2}s^2{}_\gamma\right)\right]c_{2\omega}$$
$$+ \frac{5}{96}J_3^2 s^4{}_\gamma\epsilon\left(1+\frac{5}{4}\epsilon^2\right)c_{4\omega}-\frac{3}{56}J_4\epsilon^2\left(1-5s^2{}_\gamma+\frac{35}{8}s^4{}_\gamma\right)$$
$$- \frac{J_4}{16}s^2{}_\gamma\epsilon\left(\frac{6}{7}-s^2{}_\gamma\right)\left(9+\frac{31}{4}\epsilon^2\right)c_{2\omega}-\frac{J_4}{16}s^4{}_\gamma\epsilon\left(1+\frac{5}{4}\epsilon^2\right)c_{4\omega}$$
$$- \frac{J_3}{16}s_\gamma\epsilon^2\left(3-\frac{11}{2}s^2{}_\gamma\right)s_\omega-\frac{35}{64}J_3 s^3{}_\gamma\epsilon^2 s_{3\omega} \quad (67)$$

$$\bar{a}_3 = \frac{5}{24} J_2 s^2_\gamma \epsilon s_{2\omega} - \frac{J_2^2 s^2_\gamma \epsilon}{24}\left[\frac{1}{3}\left(14 - \frac{43}{4} s^2_\gamma\right) + \frac{\epsilon^2}{8}\left(11 - \frac{19}{2} s^2_\gamma\right)\right] s_{2\omega}$$
$$- \frac{5}{96} J_2^2 s^4_\gamma \epsilon \left(1 + \frac{5}{4}\epsilon^2\right) s_{4\omega} + \frac{J_4}{16} s^2_\gamma \epsilon \left(9 + \frac{31}{4}\epsilon^2\right)\left(\frac{6}{7} - s^2_\gamma\right) s_{2\omega}$$
$$+ \frac{J_4}{16} s^4_\gamma \epsilon \left(1 + \frac{5}{4}\epsilon^2\right) s_{4\omega} - \frac{J_3}{16} s_\gamma \epsilon^2 \left(3 - \frac{11}{2} s^2_\gamma\right) c_\omega$$
$$- \frac{35}{64} J_3 s^3_\gamma \epsilon^2 c_{3\omega} \quad (68)$$

$$a_4 = - \frac{J_2}{24} \epsilon^2 s^2_\gamma c_{2\omega} - \frac{J_2^2}{36} s^4_\gamma \epsilon^2 c_{2\omega}$$
$$- \frac{J_4}{280}\epsilon^4 \left(1 - 5 s^2_\gamma + \frac{35}{8} s^4_\gamma\right) + \frac{J_2^2}{24} s^4_\gamma \left(1 + \frac{13}{4}\epsilon^2\right) c_{4\omega}$$
$$- \frac{J_4}{20} s^2_\gamma \left(\frac{6}{7} - s^2_\gamma\right) \epsilon^2 \left(\frac{11}{2} + \epsilon^2\right) c_{2\omega}$$
$$- \frac{J_4}{10} s^4_\gamma \left(\frac{1}{4} + \epsilon^2 + \frac{5}{32}\epsilon^4\right) c_{4\omega} + \frac{J_3}{64} s^3_\gamma \epsilon^3 s_\omega$$
$$- \frac{3}{16} J_3 s^3_\gamma \epsilon \left(1 + \frac{3}{4}\epsilon^2\right) s_{3\omega} \quad (69)$$

$$\bar{a}_4 = \frac{J_2}{24} \epsilon^2 s^2_\gamma s_{2\omega} + \frac{J_2^2 s^4_\gamma \epsilon^2}{36} s_{2\omega} - \frac{J_2^2}{24} s^4_\gamma \left(1 + \frac{13}{4}\epsilon^2\right) s_{4\omega}$$
$$+ \frac{J_4}{20} s^2_\gamma \left(\frac{6}{7} - s^2_\gamma\right) \epsilon^2 \left(\frac{11}{2} + \epsilon^2\right) s_{2\omega}$$
$$+ \frac{J_4}{10} s^4_\gamma \left(\frac{1}{4} + \epsilon^2 + \frac{5}{32}\epsilon^4\right) s_{4\omega} + \frac{J_3}{64} s^3_\gamma \epsilon^3 c_\omega$$
$$- \frac{3}{16} J_3 s^3_\gamma \epsilon \left(1 + \frac{3}{4}\epsilon^2\right) c_{3\omega} \quad (70)$$

$$a_5 = - \frac{J_2^2}{576} s^2_\gamma \epsilon^3 \left(1 + \frac{7}{2} s^2_\gamma\right) c_{2\omega}$$
$$+ \frac{7}{96} J_2^2 s^4_\gamma \epsilon \left(1 + \frac{\epsilon^2}{2}\right) c_{4\omega} - \frac{21}{320} J_4 s^2_\gamma \epsilon^3 \left(\frac{6}{7} - s^2_\gamma\right) c_{2\omega}$$
$$- \frac{J_4}{40} s^4_\gamma \epsilon \left(2 + \frac{7}{4}\epsilon^2\right) c_{4\omega} - \frac{7}{64} J_3 s^3_\gamma \epsilon^2 s_{2\omega} \quad (71)$$

$$\bar{a}_5 = \frac{J_2^2}{576} s^2_\gamma \epsilon^3 \left(1 + \frac{7}{2} s^2_\gamma\right) s_{2\omega} - \frac{7}{96} J_2^2 s^4_\gamma \epsilon \left(1 + \frac{\epsilon^2}{2}\right) s_{4\omega} + \frac{21}{320} J_4 s^2_\gamma \epsilon^3 \left(\frac{6}{7} - s^2_\gamma\right) s_{2\omega} + \frac{J_4}{40} s^4_\gamma \epsilon \left(2 + \frac{7}{4}\epsilon^2\right) s_{4\omega} - \frac{7}{64} J_3 s^3_\gamma \epsilon^2 c_{3\omega} \quad (72)$$

$$a_6 = \frac{J_2^2}{32} s^4_\gamma \epsilon^2 c_{4\omega} - \frac{J_4}{560} s^2_\gamma \epsilon^4 \left(3 - \frac{7}{2} s^2_\gamma\right) c_{2\omega} - \frac{J_4}{560} s^4_\gamma \epsilon^2 \left(19 + \frac{7}{2}\epsilon^2\right) c_{4\omega} - \frac{J_3}{64} s^3_\gamma \epsilon^3 s_{3\omega} \quad (73)$$

$$\bar{a}_6 = -\frac{J_2^2}{32} s^4_\gamma \epsilon^2 s_{4\omega} + \frac{J_4}{560} s^2_\gamma \epsilon^4 \left(3 - \frac{7}{2} s^2_\gamma\right) s_{2\omega} + \frac{J_4}{560} s^4_\gamma \epsilon^2 \left(19 + \frac{7}{2}\epsilon^2\right) s_{4\omega} - \frac{J_3}{64} s^3_\gamma \epsilon^3 c_{3\omega} \quad (74)$$

$$a_7 = \frac{s^4_\gamma \epsilon^3 c_{4\omega}}{128} \left(\frac{J_2^2}{2} - \frac{9}{7} J_4\right) \quad (75)$$

$$\bar{a}_7 = \frac{s^4_\gamma \epsilon^3 s_{4\omega}}{128} \left(\frac{J_2^2}{2} - \frac{9}{7} J_4\right) \quad (76)$$

$$a_8 = -\frac{J_4}{896} s^4_\gamma \epsilon^4 c_{4\omega} \quad (77)$$

$$\bar{a}_8 = \frac{J_4}{896} s^4_\gamma \epsilon^4 s_{4\omega}. \quad (78)$$

In the next section attention will be directed toward the secular equations (34 to 38).

IV. SOLUTION OF THE SECULAR EQUATIONS

These non-linear equations demonstrate an interesting interaction when the inclination angle γ is in the neighborhood of 63.4°. In this case, the term $J_2(2 - \frac{5}{2} s^2_\gamma)$ in Eq. 35 no longer dominates the behavior of ω, and there is a strong interaction with Eq. 36. If we write $\gamma = \tan^{-1} 2 + J_2\eta$ and ignore terms of the third order or greater in Eqs. 35, 37 and 38 we obtain

$$\omega' = -\frac{J_2^2}{5}\left[\left(2+\frac{6}{5}\frac{J_4}{J_2^2}\right)+\frac{\epsilon^2}{2}\left(\frac{1}{6}+\frac{81}{35}\frac{J_4}{J_2^2}\right)\right]-2J_2^2\eta$$
$$-J_2^2\left[\frac{1}{15}\left(1-\frac{18}{35}\frac{J_4}{J_2^2}\right)+\frac{\epsilon^2}{2}\left(\frac{1}{2}-\frac{69}{175}\frac{J_4}{J_2^2}\right)\right]c_{2\omega} \quad (79)$$

$$\epsilon' = -\frac{J_2^2\epsilon(1-\epsilon^2)}{15}\left(1-\frac{18}{35}\frac{J_4}{J_2^2}\right)s_{2\omega}, \quad (80)$$

$$\frac{h'}{h} = \frac{J_2^2\epsilon^2}{15}\left(1-\frac{18}{35}\frac{J_4}{J_2^2}\right)s_{2\omega}, \quad (81)$$

and if $\frac{J_4}{J_2^2}$ is not close to $\frac{35}{18}$, the dominant term for η' is given by

$$\eta' = \frac{J_2\epsilon^2}{30}\left(1-\frac{18}{35}\frac{J_4}{J_2^2}\right)s_{2\omega}. \quad (82)$$

The only possible equilibrium positions of this slow motion for $\epsilon \neq 0$ corresponds to $s_{2\omega} \equiv 0$. In this case ϵ, h and η are constants, where η has the precise value which makes $\omega' \equiv 0$. Now Eqs. 80 and 81 may be combined with Eq. 82 and ϵ and h solved exactly as functions of η. These solutions justify, to our order of approximation, treating ϵ and h as constants in the solution of Eq. 79 combined with Eq. 82. Let

$$\left.\begin{aligned}\omega' &= a + b\eta + cc_{2\omega}\\ \eta' &= ds_{2\omega}\end{aligned}\right\} \quad (83)$$

where

$$\left.\begin{aligned}a &= -\frac{J_2^2}{5}\left[\left(2+\frac{6}{5}\frac{J_4}{J_2^2}\right)+\frac{\epsilon_o^2}{2}\left(\frac{1}{6}+\frac{81}{35}\frac{J_4}{J_2^2}\right)\right]\\ b &= -2J_2^2\\ c &= -J_2^2\left[\frac{1}{15}\left(1-\frac{18}{35}\frac{J_4}{J_2^2}\right)+\frac{\epsilon_o^2}{2}\left(\frac{1}{2}-\frac{69}{175}\frac{J_4}{J_2^2}\right)\right]\\ d &= \frac{J_2\epsilon_o^2}{30}\left(1-\frac{18}{35}\frac{J_4}{J_2^2}\right)\end{aligned}\right\} \quad (84)$$

and $J_2 = J_2\left(\frac{R_e\mu}{h_e^2}\right)^2$.

The two cases of equilibrium correspond to $\omega = \pm\frac{\pi}{2}$ and $\omega = 0, \pi$. The stability of the equilibrium positions is decided by linearizing

Eq. 83 in the neighborhood of an equilibrium point. We find that $\omega = \pm \frac{\pi}{2}$ are stable equilibrium positions if $J_4/J_2^2 > 35/18$, and $\omega = 0$, π are stable equilibrium positions if $J_4/J_2^2 < 35/18$. The period of the small oscillations is very long, being

$$\frac{\sqrt{30}}{2J_2^{3/2}\epsilon_0\left|1 - \frac{18}{35}\frac{J_4}{J_2^2}\right|^{1/2}} \tag{85}$$

orbital revolutions.

Returning to Eq. 83, we may solve for ω as a function of η by solving the linear equation

$$\frac{d}{d\eta}(c_{2\omega}) = -\frac{2}{d}(a + b\eta + cc_{2\omega})$$

yielding

$$c_{2\omega} = -\left(\frac{a}{c} - \frac{bd}{2c^2}\right) - \frac{b}{c}\eta + Ke^{-(2c/d)\eta}, \tag{86}$$

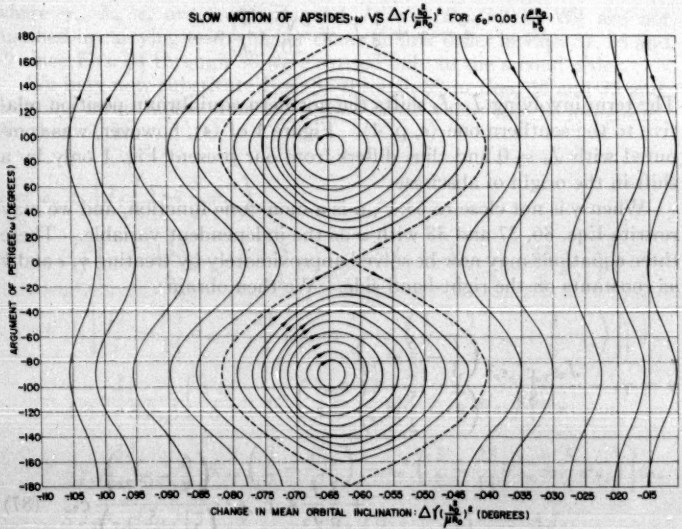

FIG. 1.

where K is the constant of integration. A family of curves of ω *versus* $\Delta\gamma\left(\frac{h_o^2}{\mu R_o}\right)^2$ is given in Fig. 1. In this case, we have chosen $J_2 = 1.637 \times 10^{-3}$, $J_4 = 1 \times 10^{-5}\left(> \frac{35}{18} J_2^2\right)$ and $\epsilon_o\left(\frac{h_o^2}{\mu R_o}\right) = 0.05$, and neglected the ϵ_o^2 terms in a, b and c. With this neglect it should be noted that for the family of curves, Eq. 86, with parameter K involves the additional parameter $\epsilon_o h_o^2$ but not ϵ_o and h_o separately.

The above phenomenon is further discussed in (4) where the interaction with "sideways drag" is also considered. It should be noted that the Δi in (4) was $i - \tan^{-1} 2$, with i signifying the average (wrt θ) of the *instantaneous* inclination angles. This average i is related to our present parameter γ by:

$$i = \gamma + \frac{1}{4} J_2 s_{2\gamma} + \frac{3}{4}\frac{J_3}{J_2}\epsilon c_\gamma s_\omega + [\text{second-order terms}].$$

In the neighborhood of $\gamma = \tan^{-1} 2$ this is essentially

$$i = \gamma + \frac{1}{5} J_2 + \frac{3\sqrt{5}}{20}\epsilon\frac{J_3}{J_2} s_\omega.$$

The term involving J_3/J_2 shifts the northern equilibrium position relative to the southern one (4, p. 4). Figure 1 of (4), however, was computed with $J_3 = 0$ and thus differs from our present Fig. 1 only by a shift in the origin of abscissae.

When γ is not close to 63.4°, ω is a monotone function, and we may rewrite Eqs. 36, 37 and 38 with ω as the independent variable. These three equations may now be solved approximately by treating γ, ϵ and h as constants on the right-hand side. We then obtain:

$$\gamma = \gamma_o - \frac{J_2 s_{\gamma_o} c_{\gamma_o}\epsilon_o^2}{8}\frac{\left(\frac{7}{3} - \frac{5}{2} s^2_{\gamma_o}\right)}{\left(2 - \frac{5}{2} s^2_{\gamma_o}\right)} c_{2\omega} + \frac{3}{8}\frac{J_4}{J_2} s_{\gamma_o} c_{\gamma_o}\epsilon_o^2 \frac{\left(\frac{6}{7} - s^2_{\gamma_o}\right)}{\left(2 - \frac{5}{2} s^2_{\gamma_o}\right)} c_{2\omega} \quad (87)$$

$$h = h_o \left[1 - \frac{J_2 s^2\gamma_o \epsilon_o^2}{8} \frac{\left(\frac{7}{3} - \frac{5}{2} s^2\gamma_o\right)}{\left(2 - \frac{5}{2} s^2\gamma_o\right)} c_{2\omega} + \frac{3}{8} \frac{J_4}{J_2} s^2\gamma_o \epsilon_o^2 \frac{\left(\frac{6}{7} - s^2\gamma_o\right)}{\left(2 - \frac{5}{2} s^2\gamma_o\right)} c_{2\omega} + \frac{3}{4} \frac{J_3}{J_2} s_{\gamma_o} \epsilon_o s_\omega \right] \quad (88)$$

$$\epsilon = \epsilon_o \left[1 + \frac{J_2}{8} s^2\gamma_o \frac{\left[\frac{1}{3} - \epsilon_o^2 \left(\frac{7}{3} - \frac{5}{2} s^2\gamma_o\right)\right]}{\left(2 - \frac{5}{2} s^2\gamma_o\right)} c_{2\omega} - \frac{3}{8} \frac{J_4}{J_2} \frac{s^2\gamma_o (1 - \epsilon_o^2) \left(\frac{6}{7} - s^2\gamma_o\right)}{\left(2 - \frac{5}{2} s^2\gamma_o\right)} c_{2\omega} \right] \quad (89)$$

where γ_o, h_o, ϵ_o are constants and $J_i = J_i(R_o\mu/h_o^2)^i$. We are not justified in carrying terms higher than the first order in Eqs. 87, 88 and 89 since Eqs. 34 through 38 were carried only to the second order.

We may now substitute Eqs. 87 through 89 in Eqs. 34 and 35 and, dropping terms higher than the second order, these may be integrated. We then obtain

$$\begin{aligned} \omega = \omega_o + \Bigg\{ & J_2 \left(2 - \frac{5}{2} s^2\gamma_o\right) \\ & + \frac{J_2^2}{2} \left[8 - \frac{133}{6} s^2\gamma_o + \frac{335}{24} s^4\gamma_o + \frac{\epsilon_o^2}{6} \left(7 - \frac{9}{2} s^2\gamma_o - \frac{45}{8} s^4\gamma_o\right) \right] \\ & + J_4 \left[\frac{12}{7} - \frac{93}{14} s^2\gamma_o + \frac{21}{4} s^4\gamma_o + \frac{27}{2} \epsilon_o^2 \left(\frac{1}{7} - \frac{1}{2} s^2\gamma_o + \frac{3}{8} s^4\gamma_o\right) \right] \Bigg\} \theta \\ & - \frac{J_2}{4(4 - 5s^2\gamma_o)} \left[\frac{s^2\gamma_o}{3} - \epsilon_o^2 \left(\frac{7}{3} - \frac{17}{2} s^2\gamma_o + \frac{25}{4} s^4\gamma_o\right) \right] s_{2\omega} \\ & + \frac{3J_4}{2J_2(4 - 5s^2\gamma_o)} \left[s^2\gamma_o \left(\frac{3}{7} - \frac{s^2\gamma_o}{2}\right) - \epsilon_o^2 \left(\frac{3}{7} - \frac{23}{14} s^2\gamma_o + \frac{5}{4} s^4\gamma_o\right) \right] s_{2\omega} \\ & + \frac{5}{4} \frac{J_2 s^2\gamma_o c^2\gamma_o \epsilon_o^2}{(4 - 5s^2\gamma_o)^2} \left[\frac{7}{3} - \frac{5}{2} s^2\gamma_o - 3 \frac{J_4}{J_2^2} \left(\frac{6}{7} - s^2\gamma_o\right) \right] s_{2\omega} - \frac{3}{4} \frac{J_3}{J_2} s_\gamma \epsilon c_\omega \end{aligned} \quad (90)$$

$$\Omega = \Omega_o + \left\{ - J_2 c_{\gamma_o} - \frac{J_2^2 c_{\gamma_o}}{2}\left[3 - \frac{13}{3}s^2_{\gamma_o} + \frac{\epsilon_o^2}{3}\left(1 + \frac{5}{4}s^2_{\gamma_o}\right)\right] - \frac{3}{2}J_4 c_{\gamma_o}\left(1 + \frac{3}{2}\epsilon_o^2\right)\left(\frac{4}{7} - s^2_{\gamma_o}\right)\right\}\theta + \frac{J_2 c_{\gamma_o}\epsilon_o^2}{2(4 - 5s^2_{\gamma_o})}\left[-\frac{7}{6} + \frac{5}{2}s^2_{\gamma_o} + 3\frac{J_4}{J_2^2}\left(\frac{3}{7} - s^2_{\gamma_o}\right)\right]s_{2\omega} + \frac{5}{4}\frac{J_2 s^2_{\gamma_o} c_{\gamma_o}\epsilon_o^2}{(4 - 5s^2_{\gamma_o})^2}\left[-\frac{7}{3} + \frac{5}{2}s^2_{\gamma_o} + 3\frac{J_4}{J_2^2}\left(\frac{6}{7} - s^2_{\gamma_o}\right)\right]s_{2\omega} \quad (91)$$

where ω_o and Ω_o are constants.

The five constants γ_o, ϵ_o, h_o, ω_o, Ω_o carry the burden of enabling the total solution (these secular equations adjoined to the results of the previous sections) to be expressed in the form which we have adopted. It should be observed that, in general, the functions γ, ϵ, h, ω and Ω do not have the values γ_o, ϵ_o, h_o, ω_o, Ω_o when $\theta = 0$, although the form of Eqs. 79 through 83 could have been chosen so that this is the case. Similarly, in the previous section the functions a_1, $\bar{a}_1$, b_1, $\bar{b}_1$ were solutions of certain "secular" differential equations, and their form could have been chosen such that they assume the value zero when $\theta = 0$. We have not done this simply because it would have complicated the appearance of a number of other terms (for example, d_1, $\bar{d}_1$, d_2, etc.). As far as the authors can see, this procedure gains nothing.

In order to obtain a complete second-order theory, it is clear that it is necessary to carry the secular differential equations to the third order.

It is interesting to interpret geometrically the major effects of the J_3 terms. It has been estimated by Kozai (5) that

$$J_3 = -2.2 \times 10^{-6} \quad (92)$$

for the planet Earth. The negative value of J_3 implies a pear-shaped Earth with a bulge in the southern hemisphere. The plane P determined by Ω and γ will always contain, as it rotates, the major portion of this bulge on the same side, provided the orbit is not polar. Now the presence of the term $\frac{3}{4}(J_3/J_2)c_\gamma \epsilon c_\theta$ in the expression for σ implies that the orbit is tilted with respect to P so that the perigee remains on the side toward the bulge and the apogee remains on the opposite side. This tilt then follows the perigee as it precesses. An observer who calculates a best-fitting plane to the orbit would interpret this as a sinusoidal fluctuation of the inclination angle of period 2π in ω.

The term $-\frac{3}{4}(J_3/J_2)s_\gamma(1 - \epsilon^2)s_{\theta+\omega}$ in the expression for ρ, shifts the orbit in the direction of the north-south apexes, so that the satellite comes closer to the center of mass in the northern hemisphere and farther

away in the southern hemisphere (the one containing the bulge). If an observer calculates a best-fitting ellipse to the orbit, then as the line of apsides rotates he would interpret this effect as a sinusoidal fluctuation of the eccentricity of period 2π in ω such that the perigee distance is least when it is at the northern apex.

There is a small but interesting effect due to a north-south balance of forces. Intuitively, due to the rotational symmetry, it is clear that we can have a circular orbit with center on the rotational axis and lying in a plane perpendicular to this axis, provided the latitude of the plane is adjusted so that the north-south forces just balance for points on the orbit. If we set $\epsilon_o = \gamma_o = 0$ in our equations, then we obtain $\sigma = (3/2)J_3$. Using the value of J_3 given by Eq. 92, it is found that a near-Earth orbit of this type would remain about 70 feet south of the equatorial plane. In this paper the equatorial plane has been defined as that plane through the center of mass perpendicular to the rotational axis. An observer on the ground whose plumb bob is perpendicular to the rotational axis would find this orbit slightly to his north.

If J_3 is second order compared with J_2, as we have assumed in this paper, then we should be able to approximate the solution up to the second order for at least several revolutions without the J_3 terms. For example, suppose we set $J_3 = 0$ everywhere that it occurs in our solution and we let Ω_o^*, ϵ_o^*, h_o^*, γ_o^* and ω_o^* be the five constants in Eqs. 87–91. Also denote the variable θ by θ^* in this solution. We will suppose for the moment that ϵ_o, s_{γ_o} are not too small, and we choose

$$\Omega_o^* = \Omega_o - \frac{3}{4}\frac{J_3}{J_2}\frac{c_{\gamma_o}\epsilon_o}{s_{\gamma_o}} c_{\omega_o},$$

$$\epsilon_o^* = \epsilon_o - \frac{3}{4}\frac{J_3}{J_2} s_{\gamma_o}(1 - \epsilon_o^2) s_{\omega_o},$$

$$h_o^* = h_o\left(1 + \frac{3}{4}\frac{J_3}{J_2} s_{\gamma_o}\epsilon_o s_{\omega_o}\right),$$

$$\gamma_o^* = \gamma_o + \frac{3}{4}\frac{J_3}{J_2} c_{\gamma_o}\epsilon_o s_{\omega_o},$$

$$\omega_o^* = \omega_o - \frac{3}{4}\frac{J_3}{J_2}\frac{c_{\omega_o}}{\epsilon_o s_{\gamma_o}}(s^2_{\gamma_o} - c^2_{\gamma_o}\epsilon_o^2). \tag{93}$$

Now if we select the correspondence

$$\theta^* = \theta + \frac{3}{4}\frac{J_3}{J_2}\frac{s_{\gamma_o}}{\epsilon_o}(1 - \epsilon_o^2) c_{\omega_o} \tag{94}$$

in the neighborhood of $\theta = 0$, then the reader may verify that the two

solutions agree up to the second order in radial distance, right ascension, and declination.

When either the eccentricity or s_γ is small, the representation of the motion without the J_3 terms can be troublesome. Suppose, for example, we consider a polar orbit with $\epsilon_o = 0$. In this case, our solution gives an orbit which closes after one revolution. Physically, there is a balance in the gravitational attraction such that an orbit which cuts the north polar axis at right angles also cuts the south polar axis at right angles. Without the J_3 terms, a good approximation is obtained by choosing the perigee on the north polar axis and $\epsilon_o^* = (-3/4)(J_3/J_2)$. However, the rotation of the line of apsides causes the perigee to move away, but the situation is actually static.

A similar case arises when $\gamma_o = 0$ but ϵ_o is rather large. The perigee, in this case, remains at the southern apex of the orbit as the perigee precesses in right ascension. Without the J_3 terms, a good approximation is obtained by choosing the inclination angle $\gamma_o^* = (-3/4)(J_3/J_2)\epsilon_o$ and the perigee at the southern apex. However, the rotation of the line of apsides causes the perigee to move toward the northern hemisphere when actually the perigee remains at a constant southern latitude.

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ELECTRIFICATION OF CARBON BLACK BY CONTACT WITH A METAL SURFACE

BY

CHARLES J. BRASEFIELD[1]

ABSTRACT

A measured quantity of carbon black was allowed to slide down a steep surface of nickel (presumably oxidized) which was electrically grounded and the charge acquired by the carbon black was determined. Blacks were selected whose oxygen content varied from 0.4 to 11.6 per cent. It was found that a black of low oxygen content acquired a large positive charge and that the charge decreased with increasing oxygen content, becoming zero for 4 per cent oxygen and negative for larger oxygen contents. It thus appears that the electron affinity of carbon black increases with increasing oxygen content of the black. The positive charge acquired by a black of low oxygen content was found to decrease as the black was heated to higher temperatures, presumably because of increasing oxygen content of the black. The oxidation of certain blacks is not permanent if the temperature reaches only 400° C. but is permanent if it reaches 600° C. The oxygen content at a particular temperature ceases to increase after the first 30 seconds at that temperature. The oxidized surface of carbon black which is responsible for its high electron affinity cannot be produced by prolonged exposure of a black to atmospheres of CO, CO_2, or C_3O_2.

Cigarette smoke is usually negatively charged, but if the cigarette is heated to 300° C., the smoke is positively charged, as though the electron affinity of the tobacco had been increased by oxidation.

1. INTRODUCTION

It has been shown (1)[2] that carbon particles in the exhaust of a motor vehicle are negatively charged. This charge was detected by means of a cylindrical Faraday cage of window screening, 50 cm. in diameter and 50 cm. long. In the center of the cage was a polonium button connected directly to the input terminal of a Keithley electrometer (Model 200A, zero centered) on which the cage rested. When positive (or negative) space charge enters the cage, the potential of the button may be several volts positive (or negative) with respect to ground. During the course of the experiments with motor vehicles, cigarette smoke was frequently blown into the Faraday cage. The electrometer deflection was sometimes negative and sometimes positive, but if short fast "drags" were taken, the cigarette smoke was generally negatively charged.

In order to learn more about the process by which space charge is generated by motor vehicles and by cigarettes, it was decided to investigate the charge acquired by carbon black after sliding over a grounded metal surface. Thomas (2) has shown that a stream of fine

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

coal particles acquires a positive charge after contact with a metal surface, but if the coal is oxidized by heating in air at 350° C. for some minutes, the charge acquired is negative and its magnitude depends on the degree of oxidation of the coal. One purpose of the experiments planned with carbon black was to investigate the relationship between the oxygen content of the carbon black and the charge acquired, and, if possible, to learn something about the nature of the oxidized surface layer.

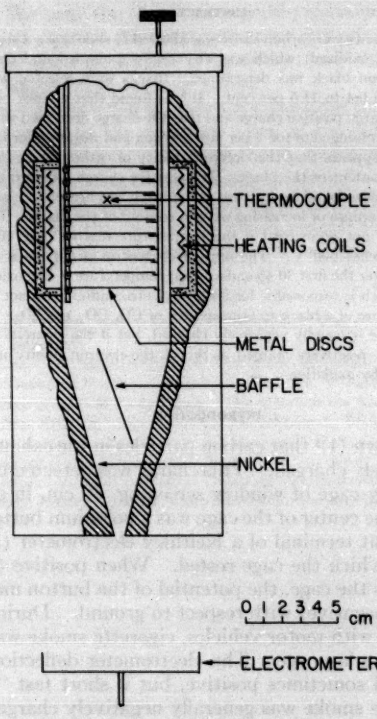

FIG. 1. Furnace, with ten metal discs inserted.

2. APPARATUS AND PROCEDURE

The experiments were carried out under conditions of pressure and humidity that existed in the laboratory. The procedure consisted simply in allowing a measured quantity of carbon black to slide down a steep surface of nickel (presumably oxidized) which was electrically

grounded. The carbon black was then collected by a shallow metal cup which rested on the input terminal of a Keithley electrometer (model 210, scale 4–0–4 volts). The electrometer deflection was proportional to the charge acquired by the carbon black. Generally, it was desired to heat-treat the carbon black before it was allowed to slide down the nickel surface. Several ways of doing this were tried and the arrangement illustrated in Fig. 1 was finally adopted. The furnace and electrometer cup were enclosed in a cage of wire screening which was grounded.

The lower end of the furnace was conical on the inside and was lined with nickel. A nickel baffle was inserted to prevent any carbon black falling through the cone without striking the walls. Samples of carbon black were placed on each of seven metal discs. The temperature of the upper three discs was considerably below average, so they were not used. The temperatures of the remaining seven discs were nearly the same, being slightly higher than the temperature indicated by the thermocouple. The discs were supported by a hinge on one side and by a pin in a vertical rod on the other side, so that ten pins were required to support the ten discs. These pins were inserted in the rod at different azimuths. If the rod was turned about 15°, the lowest disc was free to fall. Another turn of 15° would release the next disc, and so on. In this way, seven samples of carbon black could be made to slide down the nickel cone. The volume of carbon black in each sample was the same, called a standard dose. It was the amount held by a spoon 16 mm. in diameter and 3 mm. deep. The bottom of the spoon was made of window screening so that the carbon black could be dusted uniformly over a disc.

On a given day, the charges acquired by successive standard doses of the same black generally differed by less than 20 per cent. Whenever possible, the average of at least three readings was taken. The charge tended to be less when the relative humidity was high. The data here reported were taken on days of low to moderate humidity (40–60 per cent).

3. RESULTS

(*a*) *Variation of Charge with Oxygen Content of Carbon Black*

In order to investigate the relationship between the oxygen content of the carbon black and the charge acquired, a number of blacks were selected[3] whose oxygen content was known (3). A different black was placed on each of the seven discs, except one black was repeated. The charge acquired when a standard dose of the black was allowed to slide over the (oxidized) nickel surface at room temperature was measured three times on the same day. The mean charge is shown in Fig. 2 as a

[3] Kindly furnished by George E. Marsh, Manager, Special Blacks Division, Godfrey L. Cabot, Inc.

function of the oxygen content of the various blacks. There is a five-fold variation in the apparent density of these blacks, Carbolac 1 being the least dense. If account is taken of this variation in density, the charge acquired by unit mass of carbon black can be computed. This is indicated by the dashed curve of Fig. 2. It appears that the charge becomes more negative as the oxygen content of the carbon black

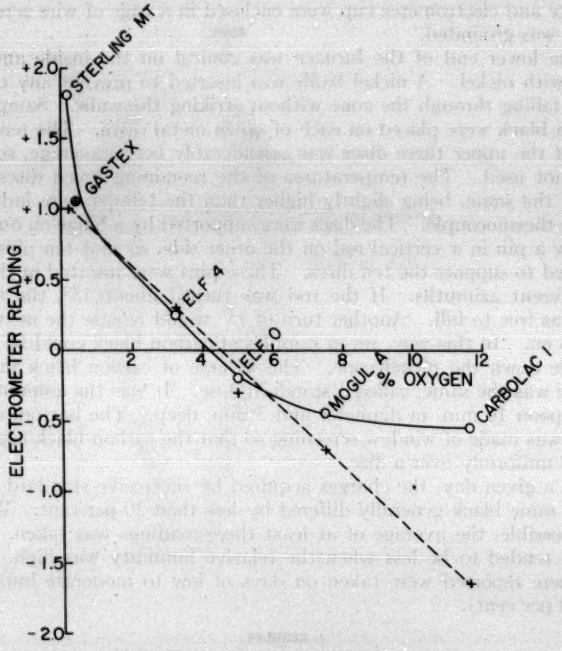

FIG. 2. Variation, with oxygen content of the black, of mean charge acquired by a standard dose of carbon black on sliding over an (oxidized) nickel surface. The charge acquired per unit mass of carbon black is indicated by the dashed curve.

increases. This is what would be expected if the work function (electron affinity) of the carbon increased with increasing oxygen content, if the same way that oxidation of nickel increases its work function (4).

If the surface over which the carbon black slides is a metal other than nickel, with a work function lower than that of nickel, the same sort of curve is obtained except that the horizontal line indicating zero electrometer reading is moved upward. For example, after sliding down

an aluminum chute, the charges acquired by unit mass of Gastex, Elf 0 and Carbolac 1 are −2.0, −2.4 and −4.2, respectively. This indicates that the horizontal line corresponding to zero electrometer reading for aluminum is to be found on Fig. 2 at an ordinate of about +2.6.

(*b*) *Variation of Charge with Particle Size*

Since the particle size generally decreases with increasing oxygen content, it might be argued that the independent variable in Fig. 2 should be particle size rather than oxygen content. This is not so, for several reasons. The particle sizes of Elf 4, Elf 0 and Mogul A are practically identical, but there is a marked difference in the charge acquired by a standard dose. Furthermore, Supercarbovar and Carbolac 46 have about equal particle diameters (about half that of Elf 0) but the mean charge for Supercarbovar (low in oxygen) was +0.7 while the charge for Carbolac 46 (high oxygen content) was −0.6. Elftex 5 has about the same oxygen content as Gastex but the particle diameter of Elftex is less than half that of Gastex. Nevertheless, the charges acquired by unit mass of Elftex and Gastex are essentially the same. It must be concluded that if particle size has any effect on the charge acquired by a standard dose of carbon black, the effect is of secondary importance.

(*c*) *Variation of Charge with Heat Treatment*

In order to investigate the nature of the oxidized surface layer, experiments were undertaken to see whether the oxygen content of a carbon black could be increased by raising the temperature of the black for certain periods of time in the same way that Thomas (**2**) was able to increase the oxygen content of fine coal. Standard doses of Gastex were placed on the seven discs, the furnace heated to a maximum temperature of 400° C., and then allowed to cool. The charge acquired when the Gastex was dumped at various temperatures, both when the temperature was increasing and when it was decreasing, is shown in Fig. 3. It appears that the electron affinity (and presumably the oxygen content) of Gastex increases with increasing temperature, becoming nearly constant at 400° C., at which temperature the electron affinity of the (oxidized) Gastex is identical with that of the (oxidized) nickel surface over which the Gastex slid.[4] However, the oxygen content of the Gastex is not permanently increased, since the charge acquired by a standard dose at 70° and at 50° is about the same for falling temperatures as for rising temperatures. If, on the other hand, the heating is carried to 600° C., then the oxidation of the Gastex ap-

[4] Bosworth (4) reports that the work function of nickel is 5.0 ev. and that of nickel oxide is 6.4 ev.

pears to be permanent (4–5 per cent); in this case, on cooling to room temperature, the charge acquired is slightly negative. Sterling MT is also permanently oxidized after heating to 600° C.

Figure 3 is interesting for another reason. It might be argued that the change in electron affinity of the carbon black with rising temperature was due to the loss of a surface layer of water molecules, rather than to an increase in oxygen content. If this were the case, however, the curve for falling temperatures should be quite different from the curve for rising temperatures. Furthermore, tests show that thin layers of carbon black can be stored for 70 hr. in a desiccator over concentrated H_2SO_4 without appreciably changing the charge acquired by a standard dose.

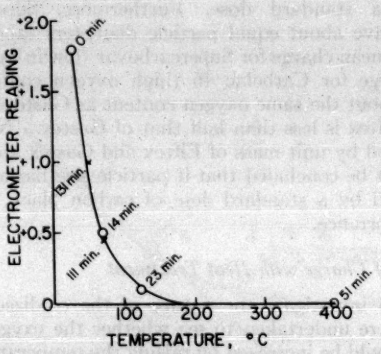

FIG. 3. Variation with temperature of the charge acquired by a standard dose of Gastex, both for increasing (*o*) and for decreasing (*x*) temperatures.

Thomas (2) found that the oxidation of coal at 350° C. increased continuously over a 15-min. period. An experiment was therefore carried out to see whether the oxidation of carbon black would increase with prolonged heating. The furnace shown in Fig. 1 was heated to 400° C. The top of the furnace was opened long enough to insert seven discs loaded with Gastex. The lowest disc was dropped about 30 seconds after insertion in the furnace, and the electrometer read −0.1 division. The remaining discs were dropped at 15-min. intervals, and the electrometer readings were +0.2, 0, −0.3, −0.1, −0.15 and −0.1, respectively. Thus the reading after 90 minutes is the same as the reading after 30 seconds, indicating that in this case, the oxidation of the carbon black was complete within 30 seconds. These samples of carbon black were saved and tested again the next day. The electrometer readings were the same as for fresh Gastex, indicating that the oxidation

at 400° C. was not permanent. However, a sample of Gastex which had been heated to 600° C. produced a negative deflection of the electrometer when tested the next day, showing permanent oxidation.

Measurements were made of the charge acquired by samples of Elf 0 at various temperatures up to 600° C. In all cases the charge was essentially the same, suggesting that the oxygen content of Elf 0 (5 per cent) is not increased by heating to 600° C.

(*d*) *Experiments with CO, CO_2 and C_3O_2*

An attempt was made to produce an oxidized surface layer on carbon black by soaking thin layers of the black for 24 hr. in an atmosphere of (*a*)[5] CO, (*b*) CO_2, and (*c*)[5,6] C_3O_2. However, the charge acquired by carbon black thus treated was not appreciably different from the charge of untreated carbon black.

4. GENERATION OF SPACE CHARGE BY MOTOR VEHICLES

To simulate conditions in the exhaust pipe of an automobile which is operating with its engine choked, Sterling MT was made to slide over a piece of rusty iron. Sterling MT is a thermal black which, because of its method of production, should be very similar to the carbon from a choked engine. A standard dose of Sterling MT, after contact with rusty iron, acquired an average charge of +0.15 unit. It therefore appears that the carbon from a choked engine is somewhat more highly oxidized than Sterling MT, since tests show the former has a negative charge (1). When the automobile engine is operated under normal conditions, any carbon in the exhaust should be still more oxidized and therefore have a greater negative charge. Since under normal conditions the exhaust is generally positive (1), it is reasonable to ascribe this charge to water droplets.

5. GENERATION OF SPACE CHARGE BY CIGARETTES

It was noted above that cigarette smoke is generally negatively charged. This can be explained by assuming that the electron affinity of the carbon particles in the smoke is greater than that of the tobacco over which the smoke passes. If the cigarette is heated to 300° C., either before or while it is being consumed, the smoke is generally positively charged, as if the electron affinity of the tobacco had been increased by oxidation. The loss of water by the cigarette on heating cannot explain the effect, because a cigarette which has been in a desiccator over concentrated H_2SO_4 for 20 hr. still produces negatively charged smoke.

[5] Kindly prepared by R. F. Trimble of the Chemistry Department of Southern Illinois University.

[6] Kindly prepared by K. A. Van Lente of the Chemistry Department of Southern Illinois University.

6. CONCLUSIONS

The charge acquired by a measured quantity of carbon black, after sliding over a nickel surface, becomes more negative as the oxygen content of the black increases, presumably because the electron affinity of the carbon black increases with increasing oxygen content. The effects observed cannot be accounted for either by differences in particle size of the blacks or by surface layers of water molecules. The oxygen content of a black can be increased up to a total of about 5 per cent by heating the black to 400–600° C. The oxidation may not be permanent if the temperature reaches only 400° C. Prolonged heating at a particular temperature does not produce any more oxidation than is produced in the first 30 seconds at that temperature. The oxidized surface of carbon black which is responsible for its high electron affinity cannot be produced by prolonged exposure of a black to atmospheres of CO, CO_2 or C_3O_2. If a cigarette is heated to 300° C., the smoke is positive instead of negative, as if the electron affinity of the tobacco had been increased by oxidation.

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ON THE ASSOCIATED FLOW RULE OF PLASTICITY BASED ON CRYSTAL SLIPS*

BY

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ABSTRACT

Associated flow rule has been commonly assumed in the limit analysis of metal structures. In this paper, aggregates of differently orientated metal crystals with finite elastic constants are considered. The coincidence of the plastic potential with yield surface of these polycrystalline aggregates is theoretically derived from the slip characteristics of single crystals. The deviation of the plastic potential from the yield surface of the aggregate increases with the elastic anisotropy of the individual crystals.

INTRODUCTION

To simplify the calculation of the ultimate load that the structure can carry, simple yield surface and its associated flow rule are commonly assumed. The present paper does not attempt to derive the yield surface under different histories of loading but aims to show the condition required of the individual crystals in the aggregate to give the coincidence of plastic potential with yield surface.

The state of stress at a material point may be represented by six independent stress components τ_{ij} of a stress tensor. Using these six stress components as coordinates in a six dimensional space, a state of stress can be represented by a point in this stress space. The region in the stress space, within which only elastic changes in strain occur, is bounded by a surface called yield surface. For work-hardening materials, when the stress point goes beyond the existing yield surface, both elastic and plastic strains are produced and a new yield surface F is formed. A similar strain space can be constructed by using the six independent strain components e_{ij} as coordinates, $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, where x_1, x_2 and x_3 are a set of rectangular coordinates and u_i is displacement along x_i-axis. The strain coordinates are superposed on the corresponding stress coordinates. In order for the dot product of the stress vector and the incremental strain vector to give incremental work, e_{11}, e_{22}, e_{33}, $2e_{12}$, $2e_{23}$ and $2e_{13}$ are used as strain coordinates. A hypoth-

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esis in plasticity states that there exists a function of stresses $g(\tau_{ij})$ such that the plastic strain increments are given by

$$de''_{ij} = \frac{\partial g}{\partial \tau_{ij}} d\lambda$$

where $d\lambda$ is a non-negative infinitesimal and double prime denotes plastic. This equation is called flow rule. A further hypothesis is made that the yield surface f and the potential function g be identical. This is called the associated flow rule of plasticity. It has been shown by Hill (1),[2] Koiter (2) and Drucker (3, 4) that these hypotheses give uniqueness of solution of stress and strain field for elastic plastic bodies. Drucker (3) has shown that this associated flow rule is satisfied for materials with positive plastic work done by the incremental stress $d\tau_{ij}de''_{ij} \geq 0$. This property $d\tau_{ij}de''_{ij} \geq 0$ does not follow directly from the present conventional laws of mechanics or thermodynamics as pointed out by Hill (5), Bland (6) and Drucker (4) and hence is not common to all materials. Since metallic structures are commonly used, it is the aim of the present paper to study the validity of this associated flow rule for polycrystalline metals.

Single crystal tests by Taylor and Elam (7, 8) have shown that under high stress, slips occur along crystallographic directions on crystal planes. These slips are the main mechanism of plastic strain. For face-centered cubic crystals such as aluminum and copper there are four crystal planes and on each plane there are three slip directions at 60 deg. apart, so there are twelve slip systems. It has been observed that single slip depends on the resolved shear stress (shear stress along the direction of slip on the crystal plane) and is independent of the normal pressure on the plane. The resolved shear stress to cause further slip is termed the critical shear stress. The slip in one system generally not only gives strain hardening, that is, raising the critical shear stress in the particular active slip system, but also gives hardening to the inactive slip systems. This is known as latent hardening. In most metals there are both active and latent hardenings. The relative amounts of these hardenings vary for different metals.

Associated Flow Rule of Plasticity for Single Crystals

Let τ_{cn} be the critical shear stress of the slip system n in the crystal, de''_{ij} a given plastic strain increment and σ_{ij} a stress which will produce this strain and σ^*_{ij} be any other stress within the yield surface. Let τ_n and τ^*_n be the resolved shear stresses of the active slip systems corresponding to σ_{ij} and σ^*_{ij}, respectively. τ_n's in the active slip systems are equal to the critical. Since σ^*_{ij} is within the yield surface, τ^*_n is

[2] The boldface numbers in parentheses refer to the references appended to this paper.

always less than the critical. Hence

$$(\tau_n - \tau^*_n)d\gamma''_n \geq 0. \tag{1}$$

The incremental work due to plastic strain increment

$$dw - dw^* = (\sigma_{ij} - \sigma^*_{ij})de''_{ij} = \sum_{n=1}^{n} (\tau_n - \tau^*_n)d\gamma''_n \geq 0. \tag{2}$$

Writing $(\sigma_{ij} - \sigma^*_{ij})$ as $d\sigma_{ij}$, the above equation gives $d\sigma_{ij}de''_{ij} \geq 0$. This is similar to Drucker's condition for stable material (9). $(\sigma_{ij}-\sigma_{ij}{}^*)$ makes an acute angle with de''_{ij}. Hence de''_{ij} is normal to the yield surface for single crystals. This has been given by Bishop and Hill (10) as the principle of maximum work which shows the coincidence of plastic potential with yield surface for single crystals.

Let y_1, y_2 and y_3 be the rectangular coordinates with one axis along the normal of the crystal plane and another axis along one direction of slip on that plane. With α, β referring to the subscripts of y's, the plastic shearing strain due to sliding on a plane with normal y_α along the direction y_β

$$\gamma''_{\alpha\beta} = \frac{\partial u_\beta}{\partial y_\alpha} = 2e''_{\alpha\beta} \tag{3}$$

where u_β is the displacement along y_β-direction. This sliding system is here called $\alpha\beta$-slip system. With $l_{\alpha i}$ and $l_{\beta j}$ denoting the cosines of the angles between y_α and x_i; y_β and x_j, respectively, the incremental strain components in x-coordinates due to $de''_{\alpha\beta}$,

$$de''_{ij} = (l_{i\alpha}l_{j\beta} + l_{i\beta}l_{j\beta})de''_{\alpha\beta}. \tag{4}$$

Referring to strain coordinates defined as $e_{11}, \cdots 2e_{12} \cdots$

For $i = j$ like $de''_{11} = 2(l_{1\alpha}l_{1\beta})de''_{\alpha\beta}$

For $i \neq j$ like $2de''_{12} = 2(l_{1\alpha}l_{2\beta} + l_{2\alpha}l_{1\beta})de''_{\alpha\beta}$.

There are as many $de''_{\alpha\beta}$'s as the active slip systems. de''_{ij} is caused by all the $de''_{\alpha\beta}$'s in the crystal. The resolved shear stress in the $\alpha\beta$-slip system due to stress tensor τ_{ij} to be critical $\tau_{\alpha\beta_c}$ may be expressed as (11)

$$\tau_{\alpha\beta} = l_{\alpha i}l_{\beta j}\tau_{ij} = \tau_{\alpha\beta_c} \tag{5}$$

where the repetition of subscripts i and j denotes summation from 1 to 3. Writing this out in long form

$$\tau_{\alpha\beta} = l_{\alpha 1}l_{\beta 1}\tau_{11} + l_{\alpha 2}l_{\beta 2}\tau_{22} + l_{\alpha 3}l_{\beta 3}\tau_{33} + (l_{\alpha 1}l_{\beta 2} + l_{\alpha 2}l_{\beta 1})\tau_{12} + (l_{\alpha 1}l_{\beta 3} + l_{\alpha 3}l_{\beta 1})\tau_{13} + l_{\alpha 2}l_{\beta 3} + l_{\alpha 3}l_{\beta 2})\tau_{23}.$$

This is a plane equation in stress space with τ_{ij}'s as coordinates and is called a yield plane. The normal to this yield plane has direction cosines with τ_{ij}-coordinate proportional to the τ_{ij} coefficient in Eq. 5. It is seen from Eq. 4, de''_{ij} is proportional to the same coefficient and hence normal to the yield plane. This is essentially the same as given by Bishop (12) showing single crystal yield surface consisting of yield planes which coincide with the plastic potential.

With the consideration of elastic strain (13), the incremental stress-strain relationship of single crystals is uniquely determined.

Associated Flow Rule of Polycrystals of Infinite Elastic Modulus

A piece of metal consists of numerous crystals of various orientations. Consider a piece of unit cube subject to a uniform surface stress S_{ij}, which is referred to as the macroscopic stress. The macroscopic strain increment is defined as

$$dE_{ij} = \tfrac{1}{2} \int (l_i du_j + l_j du_i) dS \tag{7}$$

where S is the area on the cube, l_i is the cosine of the normal to dS with x_i-axis. The work done by the surface force S_{ij} must equal the energy stored and that dissipated in the unit cube

$$S_{ij} dE_{ij} = \int (\tau_{ij} de_{ij}) dV. \tag{8}$$

Let single prime denote the elastic part

$$S_{ij}(dE'_{ij} + dE''_{ij}) = \int \tau_{ij}(de'_{ij} + de''_{ij}) dV. \tag{9}$$

For infinite elastic modulus dE'_{ij} and de'_{ij} approach zero, the above gives

$$S_{ij} dE''_{ij} = \int (\tau_{ij}) de''_{ij}) dV. \tag{10}$$

For given dE''_{ij}, S_{ij} is on the yield surface. Let S^*_{ij} be any stress vector within the yield surface, hence giving resolved shear stresses in all crystals in the cube less than the critical.

$$S^*_{ij} dE''_{ij} = \int (\tau^*_{ij} de''_{ij}) dV \tag{11}$$

$$(S_{ij} - S^*_{ij}) dE''_{ij} = \int [(\tau_{ij} - \tau^*_{ij}) de''_{ij}] dV \geq 0. \tag{12}$$

This shows the validity of the associate flow rule for polycrystals of infinite elastic modulus. Bishop and Hill (10) derived the same equation with S_{ij} denoting $\int \tau_{ij} dA$ where dA is integrated over a unit area of any plane section. In their derivation no correlation between any component of the microscopic stress and any component of displacement over any plane section of unit area is assumed. With the present definition of S_{ij} as the uniform stress over the faces of the unit cube, this assumption is not needed.

For actual polycrystals with finite elastic modulus, $S_{ij}dE'_{ij}$ may not equal $\int \tau_{ij} de'_{ij} dV$ and Eqs. 10 to 12 do not follow from Eq. 9. Hence this proof is not applicable to actual polycrystals with finite elastic modulus.

Associated Flow Rule for Polycrystals of Finite Elastic Modulus

Actual crystals have finite elastic constants. For an aggregate of elastically isotropic crystals, any change of stress within the yield surface produces only elastic strain increment which is isotropic and homogeneous and same for all crystals. The orientations of crystals in an aggregate are taken to be uniformly distributed in all possible planes and directions. The slip systems of all crystals cover all possible directions on all planes. The initial critical shear stresses are taken to be the same for all crystals, hence a loading in stress space giving critical shear stress on any plane along any direction gives a point on the initial yield surface. Therefore the initial yield surface is a surface of maximum shear stress. This has been shown by Lin (14) as

$$J_3' = -\frac{4}{27}(4K^2 + J'_2)(K^2 + J'_2)$$

where J'_2 and J'_3 are the deviated stress invariants and K is a constant. When the stress vector reaches the initial yield surface, one slip system of the most favorably orientated crystal reaches critical shear stress and is ready to slide. Since the stress is uniform before sliding, the tangent plane to the yield surface of the aggregate at this point coincides with the yield plane of this slip system of this crystal. Infinitesimal sliding in this crystal occurs while other crystals remain elastic. Macroscopic plastic strain increment is due to the plastic strain increment of this crystal alone. This plastic strain increment of this crystal is perpendicular to the yield plane, hence dE''_{ij} of the aggregate is normal to its initial yield surface. This shows that the initial yield surface coincides with the plastic potential.

After sliding occurs in this most favorably orientated crystal, the stresses in the aggregate are affected. With body force F_i per unit volume, the equilibrium condition (11) gives

$$\tau_{ij,j} + F_i = 0 \qquad (13)$$

where the subscript after comma denotes differentiation. The elastic strain equals total strain minus plastic strain, the plastic volumetric strain is taken to be zero.

$$\tau_{ij} = \lambda\delta_{ij}\theta + 2\mu(e_{ij} - e''_{ij}) \tag{14}$$

where δ_{ij} is the Kronecker delta. λ and μ are Lame's constants and θ is the volumetric strain. Substituting Eq. 14 into Eq. 13,

$$\lambda\delta_{ij}\theta_{,j} + 2\mu(e_{ij,j} - e''_{ij,j}) + F_i = 0. \tag{15}$$

It is seen that $-2\mu e''_{ij,j}$ is equivalent to F_i in the strain field equation. Hence the strain field produced by slips in crystals is same as that caused by the equivalent body forces. From the strain field, the stress field is obtained. In a fine grained metal, the stress field due to $e''_{ij,j}$ in an interior crystal can be approximated by replacing $e''_{ij,j}$ by $F_i = -2\mu e''_{ij,j}$ in an infinite medium. The stresses due to an applied force F_i in an infinite medium have been shown by Kelvin (11) to be inversely proportional to the square of the distance from the point of the application of the force, the stress due to slip dies off rapidly with the distance from the slipped crystal and the main part of the aggregate is unaffected. Most crystals will still yield when the applied aggregate stress gives critical shear stress in one slip system of any of these crystals. Hence the yield surface is deformed in the vicinity where loading proceeds. Local expansion of yield locus in the initial stage of loading beyond elastic limit has been observed by Marin and Hu (15) and others. As loading proceeds, more crystals reach critical shear stress $\tau_{\alpha\beta} = \tau_c$. To keep these crystals from further slip, $\tau_{\alpha\beta}$ must be kept not to exceed τ_c. Hence $d\tau_{\alpha\beta}$ must be less than zero. $d\tau_{\alpha\beta} = 0$ for each of these crystals gives a boundary plane to the yield surface. These planes pass through the existing stress point and form a corner in the yield surface. In a combined tension and torsion test with S_{11} denoting the applied aggregate tensile stress and S_{12} the aggregate torsional stress, the specimen is stretched beyond the elastic limit. Within the existing yield surface, all crystals are subject to the stress increments applied to the aggregate $d\tau_{ij} = dS_{ij}$, although the total stress components may vary from one crystal to another due to slips. With $\alpha\beta$ denoting a slip system reaching the critical value in some crystal, the boundary planes of the yield surface are given by

$$d\tau_{\alpha\beta} = l_{\alpha 1}l_{\beta 1}dS_{11} + (l_{\alpha 1}l_{\beta 1} + l_{\alpha 2}l_{\beta 1})dS_{12} = 0. \tag{16}$$

This gives

$$\frac{dS_{11}}{dS_{12}} = -\frac{l_{\alpha 1}l_{\beta 2} + l_{\alpha 2}l_{\beta 1}}{l_{\alpha 1}l_{\beta 1}}. \tag{17}$$

This yield locus is similar to that predicted by the slip theory of Batdorf and Budiansky (16). However in the present paper, the stresses due to slips in the crystals are included so $\tau_{ij} \neq S_{ij}$, while the slip theory assumes all crystals being subject to the same stress with $\tau_{ij} = S_{ij}$.

Among all the $\alpha\beta$ active slip systems of all the sliding crystals under tension, one gives maximum ratio of dS_{11}/dS_{12} and one gives minimum. These two values of dS_{11}/dS_{12} give the yield locus near the stress point. As stretching increases, more crystals slide, the range of $\alpha\beta$'s giving $\tau_{\alpha\beta} = \tau_c$ increases and hence the range of dS_{11}/dS_{12} given by Eq. 17 increases, the maximum value increases and the minimum value decreases. This indicates that theoretically as tension proceeds, the corner in the yield locus becomes sharper. This agrees well with the test data obtained by Naghdi, Essenberg and Koff (17). Tests by Budiansky, Dow, Peters and Shepherd (18) on thin-walled aluminum alloy cylinder in axial compression and then with additional combined compression and torsion show that the yield locus has a rather high curvature or sharp corner at the start of twisting. Here the yield surface is defined as the limiting surface in stress space, beyond which, incremental plastic strain, no matter how small, will occur. However in experiments, the yield locus is only observed when the incremental plastic strain reaches measurable magnitude. With improving accuracy in strain measurement, the corner in yield locus should be more apparent.

After sliding starts in the aggregate, the stresses in the individual crystals consist of the uniform aggregate stress S_{ij} and the non-uniform stress due to plastic slip denoted by τ_{ij_s}. Let the resolved shear stress in $\alpha\beta$-slip system corresponding to the aggregate stress S_{ij} be τ_{ij_A} and that due to slips be $\tau_{\alpha\beta_s}$. The initial critical stress of a crystal is denoted by τ_{c_0} and the increase of critical shear stress due to hardening by $\Delta\tau_c$

$$\tau_{\alpha\beta_A} = l_{\alpha i} l_{\beta j} S_{ij}.$$

In the active slip systems,

$$\tau_{\alpha\beta} = l_{\alpha i} l_{\beta j} S_{ij} + \tau_{\alpha\beta_s} = \tau_{c_0} + \Delta\tau_c. \tag{18}$$

After the stress vector goes beyond the initial yield surface, a new yield surface is formed. Within this new yield surface the incremental stress produces no plastic incremental strain and the stress increment is the same for all crystals. $\Delta\tau_{\alpha\beta} = l_{\alpha i} l_{\beta j} \Delta S_{ij}$. The limiting stress surface in the coordinates of S_{ij} giving no plastic strain increment in all crystals, is the yield surface. The stress giving critical shear stress to a slip system in a crystal gives a boundary plane to the yield surface. Due to those crystals which have not slid previously the boundary

plane to the yield surface is

$$\tau_{\alpha\beta} = l_{\alpha i} l_{\beta j}(S_{ij} + \Delta S_{ij}) + \tau_{\alpha\beta_s} = \pm \tau_{c0} \tag{19}$$

where the negative sign denotes sliding in the opposite sense. It is seen that the initial yield planes of these crystals are shifted by the amount $\tau_{\alpha\beta_s}$. This shift is similar to kinematic hardening as proposed by Prager (19). Due to any of those crystals slid previously, the boundary plane to the yield surface is given as

$$\tau_{\alpha\beta} = l_{\alpha i} l_{\beta j}(S_{ij} + \Delta S_{ij}) + \tau_{\alpha\beta_s} = \pm (\tau_{c0} + \Delta\tau_c). \tag{20}$$

If the hardening is the same in different slip systems of the crystal, the initial yield polyhedron of the crystal is expanded by $\Delta\tau_c$ and shifted by the amount $\tau_{\alpha\beta_s}$ in the stress space. This is a combination of isotropic and kinematic hardening as suggested by Hodge (20). Hardening $\Delta\tau_c$ may be different in different slip systems. If so, the different planes of the yield polyhedron move outward by different amounts by $\Delta\tau_c$.

The initial yield surface of the aggregate is bounded by the initial yield polyhedrons of all the individual crystals. After slips occur part of these yield polyhedrons of the individual crystals are translated, part are translated and expanded as discussed. The new yield surface is bounded by these new yield polyhedrons. The initial yield surface of the aggregate does not shift nor expand as a whole, but the yield polyhedrons of the individual crystals in the aggregate shift and expand differently. This corresponds to linear piecewise plasticity which has been studied by a number of distinguished investigators (21, 22).

When the stress vector reaches a new yield surface, part of the crystals have the resolved shear stress equal to their critical values and are ready to start slide or continue slide. The remaining crystals of the aggregate have resolved shear stress less than the critical. An infinitesimal increment of applied aggregate stress may cause some of the former crystals to slide by infinitesimal amount but does not give any finite stress increment so the latter crystals remain to have resolved shear stresses less than the critical and hence do not slide. The infinitesimal slip in each of the sliding crystals gives incremental plastic strain vector normal to the yield plane of the corresponding slip system. The sum of the infinitesimal slips in those sliding crystals gives an incremental plastic strain vector lying within the cone of normals of the yield planes of these sliding crystals. Hence for the subsequent yield surface plastic potential exists and coincides with the yield surface. The relative amounts of the increments of the resolved shear stresses in different sliding crystals depend on the direction of the incremental stress vector. Hence the direction of the incremental plastic strain vector dE''_{ij} due to the sum of incremental slips of all the sliding crystals depends on the

ratios of the stress components. Possible dependence of incremental plastic strain direction on incremental stress at a corner in the yield surface was first pointed out by Drucker (3).

The above derivation is based on finite isotropic elastic constants which seems to be true for tungsten (23). Most metals have anisotropic elastic constants. The variation of elastic constants with directions is rather large for α-iron, small for aluminum (23). With the consideration of anisotropic elasticity of individual crystals, the stresses vary from point to point in the aggregate even within the elastic range. The resolved shear stress in a crystal $\tau_{\alpha\beta}$ is no longer equal to $l_\alpha d_\beta \tau_{ij}$, but the sliding in this $\alpha\beta$-slip-system still gives an dE''_{ij} normal to the plane $l_\alpha d_\beta \delta_{ij} = l_\alpha d_\beta \tau_{ij} = \tau_c$. Hence the plastic strain increment is not strictly normal to the yield surface.

Bishop and Hill (10) have shown that the associated flow rule of plasticity applies to polycrystals with infinite elastic modulus. This paper shows analytically the validity of this flow rule for polycrystals of finite isotropic elastic modulus. However plastic potential deviates from the yield surface for aggregates of crystals of anisotropic elasticity.

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ON THE PURE BENDING OF AN ELASTIC, PERFECTLY PLASTIC CURVED BAR IN THE STATE OF PLANE STRAIN*

BY

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SUMMARY

The existing nonhardening, elastic-plastic solution to the problem of the pure bending of a curved bar is found to be valid only for a limited range of geometrical parameters, when the material is assumed to be compressible. In the complete treatment of the problem presented here, regimes of Tresca's yield surface are encountered in which σ_z is no longer the intermediate principal stress, rendering the problem statically indeterminate. Depending on the parameters of the problem, four types of solutions (with mutually exclusive domains of validity) are possible in which as many as six plastic zones with different stress fields may be present. Appropriate stress distributions are derived and complete results presented for a typical stage of deformation for one type of problem. In the limiting case of a completely plastic bar, it is found that the plane components of stress, the moment and the limiting position of the elastic-plastic boundaries are exactly those predicted by the rigid plastic solution.

1. INTRODUCTION

A solution to the problem of pure bending of a curved bar of nonhardening plastic material in the state of plane strain and employing the Prandtl-Reuss stress strain relations has been given by Shaffer and House (1, 2),[2] where through the assumption of incompressibility the Mises yield condition takes on a special form which renders the problem statically determinate. Since for an incompressible material Mises' and Tresca's yield conditions (when the axial stress σ_z is the intermediate principal stress) are identical, the stress field, when Tresca's yield condition is used, is also statically determinate. Furthermore, since the functional form of the displacement for an incompressible material may be found without recourse to the stress-strain relations for this problem, as shown in (3), it follows that the complete solution for an incompressible material is the same regardless of whether Mises' or Tresca's yield condition is used.

Because the assumption of incompressibility does not alter the structure of the Tresca yield condition, the statically determinate stress field, given in (1), will still be valid in part for a compressible material,[3] provided that σ_z is the intermediate principal stress in the plastic regions

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

and remains so throughout the entire range of plastic deformation. As will be shown presently, this requirement that σ_z be the intermediate principal stress places definite restrictions (depending on the value of Poisson's ratio and the geometrical parameters of the problem), on the range of validity of the solution given by Shaffer and House (1). Hence this solution, employing Tresca's yield condition, if interpreted to hold for compressible materials, must be regarded as incomplete.

The elastic-plastic problem as described in (1) begins with yielding at the inner radius $r = a$, forming a plastic region I ($a \leqq r \leqq \rho_i$) with an outward moving elastic-plastic boundary ρ_i. As deformation continues, a second plastic region II ($\rho_o \leqq r \leqq b$) initiates at the outer radius $r = b$, with an elastic-plastic boundary ρ_o that moves inward until it meets ρ_i, as the curved bar becomes fully plastic. In Section 2 of this investigation it is shown, however, that for any choice of the ratio b/a of the outer radius to the inner radius of the bar, there exists physically admissible values of Poisson's ratio ν beyond which the state of stress in the plastic domain reaches a singular regime of the Tresca yield hexagon before the bar becomes fully plastic. Similarly, for each positive value of $0 < \nu < \frac{1}{2}$ there exists a value of the ratio b/a above which a singular regime of the yield hexagon is also reached in the range of contained plastic deformation. Hence, it is necessary to re-examine the character of the solution for the entire range of $0 < \nu \leqq \frac{1}{2}$ and for all ratios b/a. Indeed, it is found that, depending on the values of b/a and ν, four different types of solutions are possible of which the statically determinate type given by Shaffer and House (1) is only one. This solution, however, constitutes (until the singular regime is encountered) the early stages of plastic deformation for all values of ν and b/a. In later stages, the stress paths in the elastic domain enter a regular regime where σ_z is not the intermediate principal stress, thus rendering the stress field statically indeterminate. In Section 3, a stress field is derived for this regular regime and in Section 4 the solution, employing this statically indeterminate stress field, is found for one stage of the deformation and the solution for the fully plastic section is obtained as a limiting case.

2. THE FOUR TYPES OF SOLUTIONS AND THEIR REGIONS OF VALIDITY

In the state of plane strain and with reference to cylindrical polar coordinates (r, θ, z), we recall from the results given in (1) that the initial yield occurs at $r = a$, when the stresses satisfy the yield condition

$$\sigma_\theta - \sigma_r = 2k. \tag{1}$$

[3] Although in the determination of the stress field the assumption of incompressibility has not been introduced explicitly by Shaffer and House (1), a detailed examination of their work reveals this assumption is necessary if the solution presented is to be valid for all ratios of the outer to the inner radii of the curved bar.

The nonvanishing components of the stress field in the ensuing plastic domain (zone I; $a \leqq r \leqq \rho_i$) are given by

$$\frac{\sigma_r}{2k} = \log \frac{r}{a}, \tag{2a}$$

$$\frac{\sigma_\theta}{2k} = 1 + \log \frac{r}{a}, \tag{2b}$$

$$\frac{\sigma_z}{2k} = \nu \left(1 + 2 \log \frac{r}{a}\right), \tag{2c}$$

where k is the yield limit in simple shear. The stress distribution given by Eqs. 2 remains a valid solution in zone I, only as long as σ_z is the

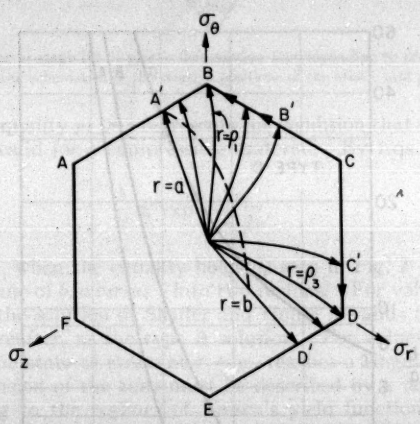

FIG. 1. Tresca's yield condition showing the stress profiles schematically. The dashed curve represents the state of stress at initial yield and the portions of the yield surface labelled A'B, B, BB', DD', D and DC' correspond to states of stress in zones I, III*a*, III*b*, II, IV*a* and IV*b*, respectively. Zone IV occurs in the type D solution only.

intermediate principal stress. An examination of (2) in principal stress space (Fig. 1), however, reveals that as the elastic-plastic boundary ρ_i moves away from $r = a$, the state of stress at $r = \rho_i$ approaches the edge (singular regime B) of the Tresca yield surface, described by

$$\text{regime } B: \begin{cases} \sigma_\theta - \sigma_r = 2k, & (3a) \\ \sigma_r = \sigma_z. & (3b) \end{cases}$$

The position $\rho_i = \rho_1$ for which this singular regime is actually reached is found by substituting Eqs. 2*a* and 2*b* (which already satisfy Eq. 3*a*) into Eq. 3*b*; hence

$$\frac{\rho_1}{a} = \exp \frac{\nu}{1 - 2\nu}. \tag{4}$$

The position $r = \rho_1$, however, is not always attained by ρ_i, since the values of b/a and ν may be such that the inward moving elastic-plastic boundary ρ_e is met first. In this event the solution in (1) is valid, and the limiting position $\rho_i = \rho_e$ is given by

$$\rho^2 = ab. \tag{5}$$

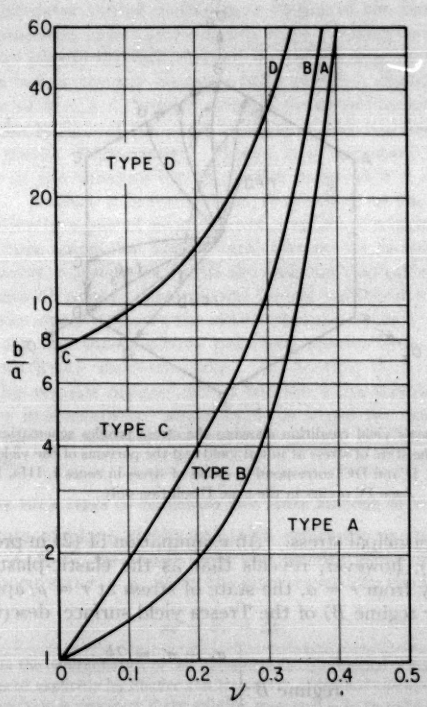

FIG. 2. Regions of validity in the b/a *versus* ν plane for each of the four problem types.

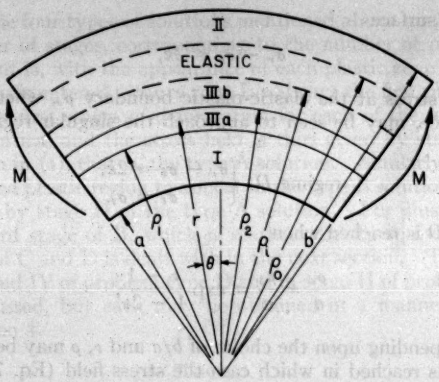

FIG. 3. Curved bar in stage III of plastic deformation (corresponding to problem types B, C, or D), showing schematically the relative positions of the elastic and plastic zones.

Thus, the inequality $\rho_1 > \rho$ represents the condition that the stress field (Eqs. 2) be valid for a compressible material. By Eqs. 4 and 5, this becomes

$$\frac{b}{a} \leqq \exp \frac{2\nu}{1-2\nu}. \tag{6}$$

A plot of (6), when the equality holds, is seen in Fig. 2 (curve OA) to divide the plane of b/a *versus* ν into two regions. For values of b/a and ν below OA the solution of Shaffer and House is valid; this will be referred to, hereafter, as the type A solution. For values of b/a and ν above OA, the state of stress at $r = \rho_1$, reaches a singular regime and further expansion of the zone must be described by a new stress field, corresponding to the regimes of Tresca's yield function encountered. The plastic zone which forms adjacent to zone I will be referred to as zone III (see Fig. 3).

In a similar manner, from an examination of the statically determinate stress field in zone II ($\rho_o \leqq r \leqq b$)

$$\frac{\sigma_r}{2k} = \log \frac{b}{r}, \tag{7a}$$

$$\frac{\sigma_\theta}{2k} = \log \frac{b}{r} - 1, \tag{7b}$$

$$\frac{\sigma_z}{2k} = \nu \left(2 \log \frac{b}{r} - 1 \right), \tag{7c}$$

lying on the surface

$$\sigma_r - \sigma_\theta = 2k, \tag{8}$$

the state of stress at the elastic-plastic boundary ρ_e, when considered in stress space, may be seen to approach the singular regime (Fig. 1) described by

$$\text{regime } D: \begin{cases} \sigma_r - \sigma_\theta = 2k, \\ \sigma_\theta = \sigma_z. \end{cases} \tag{9}$$

The regime D is reached when

$$\frac{\rho_e}{a} = \frac{\rho_2}{a} = \exp \frac{\nu - 1}{1 - 2\nu}, \tag{10}$$

but again depending upon the choice of b/a and ν, ρ may be reached by ρ_e before ρ_2 is reached in which case the stress field (Eq. 7) would remain valid throughout the problem. It will be shown later (Section 4) that Eq. 5 is valid even in the presence of a plastic zone III, so that by Eqs. 10 and 5 the criterion that the singular regime D, described by Eq. 9, never be reached is given by

$$\frac{b}{a} \leqq \exp \frac{2(1 - \nu)}{1 - 2\nu}. \tag{11}$$

Equation 11, when the equality holds, is plotted (curve CD) in Fig. 2; in the region above CD the stress at ρ_e in zone II reaches the regime D of Tresca's yield hexagon, and subsequent deformation must therefore be accompanied by the formation of a plastic zone IV ($\rho_e \leqq r \leqq \rho_2$) adjacent to zone II.

In order to classify the various types of solutions that may occur following initial yield and the formation of zone I, it is necessary to establish which of the zones II or III will appear after zone I. In the absence of zones III and IV the position of ρ_i, when initial yield occurs at $r = b$, may be obtained by placing $\rho_e = b$ in the relation between ρ_e and ρ_i

$$\left(\frac{\rho_e}{\rho_i}\right)^2 \left(1 - \log \frac{ab}{\rho_i^2}\right) = 1 + \log \frac{\rho_e^2}{ab}. \tag{12}$$

Substitution of ρ_1 from Eq. 4 into Eq. 12 then gives the value of b/a for which zones II and III appear simultaneously. A plot of the resulting curve is shown by OB in Fig. 2; above OB zone III precedes zone II, while below OB and above OA zone II precedes zone III. Further analysis reveals zones II and III always precede zone IV so that only four types of solutions are possible; these are summarized in Table I and are to be identified with the correspondingly labelled areas in Fig. 2.

Each of the four types of solutions mentioned above may be divided into a number of stages, corresponding to the number of plastic zones occurring; that is, with the appearance of each plastic zone a new stage of the problem is entered. Since zone I is always the first plastic region to appear, the first stages of deformation for each of the four solution types are identical and the stress field is thus given by stage I of the solution given in (1), that is, the type A solution. Similarly, since zone II is the second plastic region to appear in the type B solution, the stress field is given by stage II of the type A solution. For illustrative purposes the third stage of B, which is mathematically equivalent to the third stages of C and D is dealt with in the next section. The solutions in stages II and IV of problem type D and in stage II of problem type C are not discussed, but each may be obtained in a manner similar to that of Section 4.

TABLE I.

Type of Solution	Order of Appearance of Plastic Zones			
A	I	II	—	—
B	I	II	III	—
C	I	III	II	—
D	I	III	II	IV

3. THE STRESS FIELD IN ZONE III

Having established the ranges of b/a and ν for which the distributions (Eqs. 2 and 7) are no longer capable of completely describing the plastic state of stress in all stages of deformation, we now proceed to derive stress fields that are valid in the plastic region III, corresponding to the new regimes encountered by the stress paths on the Tresca yield surface. Through the use of such stress fields, the complete solution for stage II of both the C and D problem types as well as for stage III of the B, C, and D problem types may be obtained. An examination of the loading paths (Fig. 1) in stage I of the type A solution (which is common to all solution types) shows that once the stress at $r = \rho_i$ in zone I has reached the singular regime B, further yielding may occur according to

$$\sigma_\theta - \sigma_z = 2k. \tag{13}$$

The presence of σ_z in Eq. 13 renders the stress field statically indeterminate and use of the flow rule

$$\{\dot{\epsilon}_r'', \dot{\epsilon}_\theta'', \dot{\epsilon}_z''\} = \lambda \left\{ \frac{\partial f}{\partial \sigma_r}, \frac{\partial f}{\partial \sigma_\theta}, \frac{\partial f}{\partial \sigma_z} \right\}, \tag{14}$$

must be made in order to obtain the form of the stress field.

Since in the state of plane strain the total axial strain is zero, use of the flow rule (Eq. 14), together with Eq. 13, followed by an integration with respect to time, gives

$$\begin{aligned} \epsilon_r'' &= 0, \\ \epsilon_\theta'' &= -\epsilon_z'' = \epsilon_z', \\ \epsilon_z'' &= -\epsilon_z', \end{aligned} \tag{15}$$

where prime and double prime refer to the elastic and plastic components of strain, respectively. Because the plastic strains are expressible in terms of the elastic component of the axial strain, the total strains may be obtained in terms of the stresses through the generalized Hooke's law and are given by

$$\begin{aligned} \epsilon_r &= \frac{1}{E}[\sigma_r - \nu(\sigma_\theta + \sigma_z)], \\ \epsilon_\theta &= \frac{1}{E}[\sigma_\theta + \sigma_z - \nu(\sigma_\theta + 2\sigma_r + \sigma_z)], \\ \epsilon_z &= 0. \end{aligned} \tag{16}$$

Substituting in Eqs. 16 for σ_z and σ_θ from the yield condition (Eq. 13) and the stress differential equation of equilibrium

$$\sigma_\theta = \frac{\partial}{\partial r}(r\sigma_r), \tag{17}$$

respectively, the appropriate equation of compatibility then becomes

$$2(1-\nu)r^2\frac{\partial^2\sigma_r}{\partial r^2} + 6(1-\nu)r\frac{\partial\sigma_r}{\partial r} + (1-2\nu)\sigma_r = 2(1-\nu)kP, \tag{18}$$

where P is an arbitrary function of time. The solution of Eq. 18 is

$$\frac{\sigma_r}{2k} = \frac{D_1}{r^{s_1}} + \frac{D_2}{r^{s_2}} + P, \tag{19a}$$

where D_1 and D_2 are arbitrary functions of time, and

$$\left.\begin{matrix} s_1 \\ s_2 \end{matrix}\right\} = 1 \mp \frac{1}{\sqrt{2(1-\nu)}}. \tag{19b}$$

By Eq. 17 and the yield condition (13)

$$\frac{\sigma_\theta}{2k} = (1 - s_1)\frac{D_1}{r^{s_1}} + (1 - s_2)\frac{D_2}{r^{s_2}} + P, \tag{19c}$$

$$\frac{\sigma_z}{2k} = \frac{\sigma_\theta}{2k} - 1. \tag{19d}$$

It is found, however, that the stress field in zone III cannot be described completely by Eqs. 19 alone and still satisfy all of the continuity requirements at the boundaries ρ_o and ρ_i. These conditions may be satisfied however, if zone III is divided into subregions IIIa ($\rho_1 \leqq r \leqq \rho_2$) and III$b$ ($\rho_2 \leqq r \leqq \rho_i$). If the state of stress in the subregion IIIa is taken to lie at the singular regime B of the yield surface, described by Eqs. 3, then by Eq. 17,

$$\frac{\sigma_r}{2k} = \frac{\sigma_z}{2k} = \log\frac{r}{a}, \tag{20a}$$

$$\frac{\sigma_\theta}{2k} = 1 + \log\frac{r}{a}, \tag{20b}$$

the stress field in the subregion IIIb is given by Eqs. 19, while in the elastic domain the stresses are

$$\frac{\sigma_r}{2k} = \left(\frac{\rho_o}{r}\right)^2 A + B\left(1 - 2\log\frac{\rho_o}{r}\right) + 2C, \tag{21a}$$

$$\frac{\sigma_\theta}{2k} = -\left(\frac{\rho_o}{r}\right)^2 A + B\left(3 - 2\log\frac{\rho_o}{r}\right) + 2C, \tag{21b}$$

$$\frac{\sigma_z}{2k} = 4\nu\left[B\left(1 - \log\frac{\rho_o}{r}\right) + C\right], \tag{21c}$$

where A, B, and C are arbitrary functions of time.[4]

It may be noted here that the stress field in plastic zone IV of stage IV of problem type D (which like zone III also consists of two subregions) is described by states of stress lying in the regular regime ED and the singular regime D of the Tresca yield hexagon (Fig. 1). These stresses may be derived in a manner similar to that used in obtaining Eqs. 19 and 20.

[4] See, for example, Timoshenko and Goodier (4).

4. THE STRESS FIELD IN THE PRESENCE OF ZONES I, II, AND III

Since the solution for stages I and II of the problem type A corresponds identically to the first two stages of problem type B, we are motivated to complete the type B solution by considering stage III of this problem in some detail. In addition, with reference to Table I, it may be seen that the presence of zone III together with zones I and II also typifies stage III of the problem types C and D, as well as the problem type B. The form of the state of stress in the elastic and plastic zones, shown schematically in Fig. 3, may be summarized as follows: in zone I ($a \leqq r \leqq \rho_1$) the stress field is given by Eq. 2 and satisfies the yield condition (Eq. 1); in zone IIIa ($\rho_1 \leqq r \leqq \rho_2$) by Eqs. 20, satisfying Eqs. 3; in zone IIIb ($\rho_2 \leqq r \leqq \rho_i$) by Eqs. 19 satisfying Eqs. 13; in the elastic zone ($\rho_i \leqq r \leqq \rho_e$) by Eqs. 21; and in zone II ($\rho_e \leqq r \leqq b$) by Eqs. 7 which satisfy Eq. 8.

From an examination of the stress field it is seen that the seven independent conditions that ensure the continuity of the stresses at the boundaries ρ_i, ρ_e, and ρ_1 together with the end condition

$$M = -\int_a^b \sigma_\theta r dr \tag{22}$$

are not sufficient to determine all of the nine quantities A, B, C, D_1, D_2, P, M, ρ_e and ρ_1 for a given value of ρ_i (which may be selected here as the time variable, since it increases monotonically throughout the deformation). Clearly, the stress field cannot be determined completely without recourse to the displacement field which demonstrates further the statically indeterminate nature of the B, C, and D type solutions. It will be shown in Section 5 that six additional unknown functions of time are introduced in the displacement field along with seven independent continuity conditions at the boundaries ρ_e, ρ_i, ρ_1, and ρ_2. Thus, considering the stress and displacement fields together, there are fifteen equations (nonlinear in ρ_e, ρ_i, and ρ_1) in fifteen unknowns. The stress field, however, may be found to within the arbitrary function P and conclusions may be reached regarding the fully plastic solution without reference to the displacements; therefore a detailed treatment of the displacement field is left to the next section.

The coefficients (A, B, C), appearing in the solution for the elastic zone, may be completely determined in terms of the positions of the elastic-plastic boundaries (ρ_e and ρ_i) through the continuity of σ_r and σ_θ at $r = \rho_e$ and by the satisfaction of the yield condition (Eq. 13) at $r = \rho_i$, giving

$$A = \frac{1}{N}\left[-4(2-\nu) + 4(1-2\nu)\log\frac{b}{\rho_i}\right],$$

$$B = \frac{1}{N}\left[-6 + 4\nu - 2x^2 + 4(1-2\nu)\log\frac{b}{\rho_o}\right],$$

$$C = \frac{1}{N}\left\{7 - 4\nu + x^2 + \log x[4(1-\nu) - 2x^2 - 4(1-2\nu)\log x] + \log\frac{b}{\rho_i}[2x^2 - 6 + 8\nu + 4(1-2\nu)\log x]\right\}, \tag{23}$$

where

$$N = 4[-1 + x^2 + 2(1-2\nu)\log x],$$

and

$$x = \rho_o/\rho_i.$$

Since σ_z is the intermediate principal stress in zone II, its continuity at $r = \rho_o$ is ensured by requiring the continuity of σ_r and σ_θ, since for the regime ED (Fig. 1) of the Tresca yield surface $\sigma_z = \nu(\sigma_r + \sigma_\theta)$. At $r = \rho_i$, however, since σ_r is the intermediate principal stress in zone IIIb, the continuity of σ_z is not implied by that of the plane components, so that there exists three independent continuity requirements on the stresses σ_r, σ_θ and σ_z. In this connection, it should be mentioned that equilibrium and the yield conditions only require that σ_r and the difference $\sigma_\theta - \sigma_z$ be continuous at $r = \rho_i$. If, however, σ_θ and σ_z individually are allowed to be discontinuous, it is found through the compressibility equation that the displacement must also be discontinuous which is physically unacceptable. Hence, in addition to the requirement that $(\sigma_\theta - \sigma_z)_{\rho_i} = 2k$, employed above, we also require continuity in σ_r and σ_θ at $r = \rho_i$, so that by Eqs. 19 and 21

$$D_1 = \frac{\rho_i^{s_1}}{s_2 - s_1}\{-s_1x^2A + B(2 + s_2 - 2s_2\log x) + s_2(2C - P)\},$$
$$D_2 = \frac{\rho_i^{s_2}}{s_2 - s_1}\{s_2x^2A - B(2 + s_1 - 2s_1\log x) - s_1(2C - P)\}. \tag{24}$$

At $r = \rho_2$, since both of the stress fields (Eqs. 19 and 20) satisfy Eq. 13, there are only two independent stress continuity conditions, which may be imposed on σ_r and σ_θ, giving

$$D_1 = \frac{\rho_2^{s_1}}{s_2 - s_1}\left[1 + s_2\left(\log\frac{\rho_2}{a} - P\right)\right],$$
$$D_2 = \frac{-\rho_2^{s_2}}{s_2 - s_1}\left[1 + s_1\left(\log\frac{\rho_2}{a} - P\right)\right]. \tag{25}$$

Equating the values of D_1 and D_2 obtained in Eqs. 24 and 25, leads to the following two equations describing the interaction of the boundaries ρ_o, ρ_i, ρ_2:

$$\begin{aligned} -Ax^2s_1 + B(s_1 + 2) - 2s_2B\log x - Ps_2 + 2Cs_2 \\ = y^{s_1}\left(s_2\log\frac{\rho_2}{a} - s_2P + 1\right), \\ -Ax^2s_2 + B(s_1 + 2) - 2s_1B\log x - Ps_1 + 2Cs_1 \\ = y^{s_2}\left(s_1\log\frac{\rho_2}{a} - s_1P + 1\right), \end{aligned} \tag{26}$$

where $y = \rho_2/\rho_i$.

The above expressions correspond to the single relation (12) between ρ_o and ρ_i which holds on stage II of the type A solution. By setting $\rho_i = \rho_1 = \rho_2$ in Eqs. 26, where ρ_1 is given by Eq. 4, it is found that Eqs. 26 and 12 predict the same value for ρ_o. This is to be expected, since $\rho_i = \rho_1 = \rho_2$ represents the transition from stage II of a type B solution, where Eq. 12 is valid, to stage III, where Eqs. 26 hold. Allowing x to approach unity in Eqs. 26 leads to the result that

$$\lim_{x\to 1} y = 1,$$

so that in the limiting case of the fully plastic state, the plastic zone III*b* vanishes. This result is of particular significance since the stress field in this state is then given only by Eqs. 2, 7 and 20, so that the plane components of stress are seen to be identical to the plane components of stress determined by Shaffer and House (1), as the limiting case of an elastic-plastic problem (corresponding here to mode A) and identical also to the results of Hill (5), obtained directly using a rigid plastic material for the related problem of a bent sheet. Thus, by equilibrium the radial stress must be continuous in the limit so that by equating Eq. 7*a* with Eq. 20*a*, the limiting value of ρ_i, ρ_o, and ρ_1 is again given by Eq. 5. By Eq. 22 the bending moment per unit width of cross section, with the aid of Eqs. 2*b*, 7*b*, 19*c*, 20*b*, and 21*b*, is found to be

$$\begin{aligned} 4\left[\frac{2M}{ka^2} - 1 - \left(\frac{b}{a}\right)^2\right][1 - x^2 - 2(1 - 2\nu)\log x] \\ = 4\left(\frac{\rho_i}{a}\right)^2\left[(x^2 - 1)\left(1 + 2\log\frac{\rho_i}{b}\right) - 24\log x\right] \\ + 16\left(\frac{\rho_o}{a}\right)^2\log x\left[(1 - 2\nu)\log\frac{\rho_i}{b} + 2 - \nu\right] \end{aligned}$$

Continued on next page

$$+\frac{4}{s_2-s_1}\left(\frac{\rho_2}{a}\right)^2[x^2-1+2(1-2\nu)\log x]$$

$$\times\left\{\frac{1-s_1}{2-s_2}(y^{s_1-2}-1)\left(1+s_2\log\frac{\rho_2}{a}-s_2P\right)\right.$$

$$-\frac{1-s_2}{2-s_2}(y^{s_2-2}-1)\left(1+s_1\log\frac{\rho_2}{a}-s_1P\right)$$

$$\left.+\frac{s_2-s_1}{2}\left[\left(\frac{\rho_i}{\rho_2}\right)^2(1-y^2)P+2\left(1+2\log\frac{\rho_1}{a}\right)\right]\right\}. \quad (27)$$

The limiting value of M in the fully plastic stage, that is, $M = M^{**}$ may be found from Eq. 27, following one application of L'Hospital's rule, by allowing ρ_i, ρ_o, ρ_2 to approach ρ which leads to the result

$$\frac{M^{**}}{ka^2}=\frac{1}{2}\left[\frac{b}{a}-1\right]^2, \quad (28)$$

which is also in agreement with that of (1) and (5).

Since, during Stage III of types B, C, and D solutions, both zones II and III are present (see table), the results (Eqs. 23, 24, 25, 26, and 27) of this section are valid for all three problem types; the expressions for the limiting case $\rho_o/\rho_i \to 1$, however, require separate treatment for problem type D due to the appearance of zone IV before the fully plastic state is reached.

5. THE DISPLACEMENT FIELD

For the sake of completeness and in order to determine the time function appearing in the solution for the stress field, the associated displacement field is determined in this section. In zones I, II, and III*a* the displacements may be obtained through integration of the strain-displacement relations, where the strains are found through the equation of compressibility

$$\epsilon_r+\epsilon_\theta=\frac{1-2\nu}{E}(\sigma_r+\sigma_\theta+\sigma_z), \quad (29)$$

and the appropriate condition of strain compatibility

$$\frac{\partial^2\epsilon_r}{\partial\theta^2}-r\frac{\partial\epsilon_r}{\partial r}+\frac{\partial}{\partial r}\left(r^2\frac{\partial\epsilon_\theta}{\partial r}\right)=0, \quad (30)$$

where $\epsilon_{r\theta}$ has been taken equal to zero. Six arbitrary functions of time appear in the resulting expressions for u_r and u_θ. In the elastic zone and in plastic zone III*a*, since the total strains are expressible in terms

of the stresses by the generalized Hooke's law and Eq. 13, respectively, no arbitrary time function other than that due to a rigid body motion term appears in the displacements which are found to be

$$\frac{Gu_r}{kr} = -\left(\frac{\rho_o}{r}\right)^2 A - B + 2(1-2\nu)B\log\frac{r}{\rho_o} + K_o\frac{\cos\theta}{r}, \quad (31a)$$

$$\frac{Gu_\theta}{kr} = 4(1-\nu)B\theta + K_o\frac{\sin\theta}{r}, \quad (31b)$$

$$\frac{Gu_r}{kr} = \frac{\alpha_1 D_1}{r^{s_1}} + \frac{\alpha_2 D_2}{r^{s_2}} + \frac{1-2\nu}{1+\nu}P + \frac{\nu}{1+\nu} + K_1\frac{\cos\theta}{r}, \quad (32a)$$

$$\frac{Gu_\theta}{kr} = \left[\frac{1-2\nu}{1+\nu}P - \frac{1}{1+\nu}\right]\theta - K_1\frac{\sin\theta}{r}, \quad (32b)$$

$$\alpha_1 = \frac{1-2\nu(1-s_1)}{(1-s_1)(1+\nu)} = \frac{2[(1-\nu)(1-s_1)-\nu]}{1+\nu},$$

$$\alpha_1 = \frac{1-2\nu(1-s_2)}{(1-s_2)(1+\nu)} = \frac{2[(1-\nu)(1-s_2)-\nu]}{1+\nu}.$$

By the strain-displacement relations, Eq. 29 assumes the form

$$\frac{u_r}{r} = \frac{1-2\nu}{E}(\sigma_r+\sigma_\theta+\sigma_z) - \epsilon_r' - \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}, \quad (33)$$

which is valid by the first of Eqs. 15 in zone IIIa as well as the elastic zone. Since the form of u_θ is such that the continuity of u_θ implies that of $\frac{\partial u_\theta}{\partial\theta}$, it is seen from Eq. 33 that the continuity of u_r at $r=\rho_i$ is a consequence of the continuity of the stress field and need not be further imposed. The six unknowns and P may thus be found from the remaining seven continuity conditions in displacements at the boundaries ρ_1, ρ_2, ρ_i and ρ_o. Imposing these requirements, the radial component of displacement in zones I, II, and IIIa becomes, respectively

$$\frac{Gu_r}{kr} = \left(\frac{\rho_2}{r}\right)^2\left[\frac{\alpha_1 D_1}{\rho_2^{s_1}} + \frac{\alpha_2 D_2}{\rho_2^{s_2}} + \frac{1-2\nu}{1+\nu}P + \frac{14\nu-5}{4(1+\nu)}\right]$$
$$+ (1-2\nu)\left[\log\frac{r}{a} - \left(\frac{\rho_1}{r}\right)^2\log\frac{\rho_1}{a}\right]$$
$$- \frac{3}{2}\frac{1-2\nu}{1+\nu}\left[\left(\frac{\rho_2}{r}\right)^2\left(\log\frac{\rho_2}{r}-1\right) - \left(\frac{\rho_1}{r}\right)^2\left(\log\frac{\rho_1}{a}-\frac{1}{6}\right)\right]$$
$$+ 2(1-\nu)B\left[\left(\frac{\rho_2}{r}\right)^2 - 1\right] + K\frac{\cos\theta}{r}, \quad (34)$$

$$\frac{Gu_r}{kr} = -A\left(\frac{\rho_o}{r}\right)^2 + 2(1-2\nu)\left(\frac{\rho_o}{r}\right)^2 C - B + (1-2\nu)\left[\left(\frac{\rho_o}{r}\right)^2 \log\frac{\rho_o}{r} - \log\frac{r}{b}\right] + B(1-2\nu)\left[\left(\frac{\rho_o}{r}\right)^2 - 1\right] + K\frac{\cos\theta}{r}, \quad (35)$$

$$\frac{Gu_r}{kr} = \left(\frac{\rho_2}{r}\right)^2\left[\frac{\alpha_1 D_1}{\rho_2^{\alpha_1}} + \frac{\alpha_2 D_2}{\rho_2^{\alpha_2}} + \frac{1-2\nu}{1+\nu}P + \frac{14\nu-5}{4(1+\nu)}\right] - \frac{3}{2}\left(\frac{1-2\nu}{1+\nu}\right)\left\{\left(\frac{\rho_2}{r}\right)^2\left(\log\frac{\rho_2}{a} - 1\right) - \log\frac{r}{a} + \frac{1}{6}\right\} + 2(1-\nu)B\left[\left(\frac{\rho_2}{a}\right)^2 - 1\right] + K\frac{\cos\theta}{r}, \quad (36)$$

and the tangential component of the displacement in all zones including the elastic zone and IIIa is given by

$$\frac{Gu_\theta}{kr} = 4(1-\nu)B\theta - K\frac{\sin\theta}{r}, \quad (37)$$

with the particular result

$$P = 4B\left(\frac{1-\nu^2}{1-2\nu}\right) + \frac{1}{1-2\nu}. \quad (38)$$

The time function $K = K_o = K_1$ is the same for each zone and may be evaluated by constraining a point on the $\theta = 0$ axis. This is most conveniently done by setting[5] $u_r(r, \theta, t)$ equal to zero at $r = \rho$, $\theta = 0$; that is,

$$u_r(\rho, 0, t) = 0,$$

since $r = \rho$ is the last remnant of the elastic zone. Thus K may be found from Eq. 31a in all stages of the problem.

6. DISCUSSION AND RESULTS

Although the regions of validity of the A, B, C and D types solution are mutually exclusive (Fig. 2), it is of interest to consider the error introduced by regarding the type A solution as valid for all values of $0 < \nu \leqq \frac{1}{2}$ and the ratio b/a (in this connection see footnote 3 in the introduction), including the solution discussed in Section 4. For a given choice of b/a this is equivalent to assuming that Poisson's ratio lies to the right of curve OA in Fig. 2. It is found that for all practical purposes, the presence of zone III has little effect on the value of the

[5] It has already been tacitly assumed by the form of u_θ that $\frac{\partial u_\theta}{\partial r} = u_\theta = 0$ for $\theta = 0$.

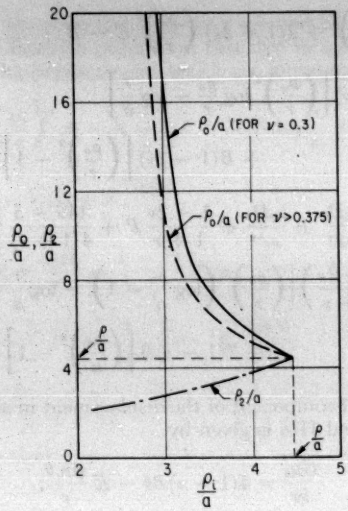

FIG. 4. Comparison of the value of ρ_0 predicted by the type C solution (shown solid) for $b/a = 20$, $\nu = 0.3$ with the value predicted by the type A solution (shown dashed) for the same value of ρ_i.

bending moment even for moderately large values of b/a; for example, for $b/a = 20$, $\nu = 0.3$, the bending moment obtained from Eq. 27 differs by less than one-tenth of one per cent from the value predicted by the type A solution which is actually only valid for $\nu > 0.375$. The predicted positions of the elastic-plastic boundaries however, differ considerably as may be seen from Fig. 4.

Acknowledgment

The author wishes to thank Professor P. M. Naghdi for the many helpful suggestions made during the course of this investigation.

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BARTOL RESEARCH FOUNDATION
MARTIN A. POMERANTZ, DIRECTOR

THE EVAPORATION OF THORIUM METAL

BY

DANIEL L. GOLDWATER AND WILLIAM E. DANFORTH

Some years ago we wished to know the rate of evaporation of bulk thorium metal. C. Zwikker in 1928 included the results of his own unpublished measurements in a review article (1).[1] Values quoted by other authors (2, 3, 4, 5, 6) for heat of evaporation of thorium metal or its rate of evaporation, seem to refer back to this work.

In the hope of quickly checking the published data we set out to measure the rate of evaporation of thorium from a tungsten dish heated in vacuuo by RF induction. Measurements were made by observing the weight loss of the specimen and also by finding the gain of weight of a nearby collector. Appropriate precautions were taken to define the evaporating area, obtain true temperature values, etc. In the earliest experiments, the thorium was exposed to total pressures somewhat above 10^{-5} mm. of Hg during the heating cycle. At temperatures around 1500° C., data from these experiments were in fair agreement with the previously published data. At lower temperatures, however, very erratic data were obtained, the specimen sometimes even showing a small gain of weight. Furthermore, it was noticed that the thorium had a golden patina when the specimen was removed from the vacuum system for weighing.

In subsequent experiments, the thorium was maintained in a vacuum always better than 2×10^{-6} mm. of Hg during the entire cycle, and care was taken that the specimen had completely cooled before the vacuum system was let down to air. The thorium was then found to keep a silvery lustre, and the evaporation data, although they still exhibited considerable scatter, were consistently 5 to 10 times higher than the previously published values. A comparison of earlier data and our data is shown in Fig. 1.

It may be presumed that the patina and the low evaporation rates obtained at the "higher" pressures were both due to surface oxidation of the thorium. Thorium oxide has a very low evaporation rate at the temperatures concerned. The slope of either set of data is entirely inconsistent with the heat of evaporation of thorium oxide (7). Thus it appears that a surface oxide forms at "higher" pressures which inhibits, but does not entirely block the evaporation of the underlying metal. Even after improving the vacuum conditions, we occasionally observed small dark specks on the thorium surface which were probably residual islands of oxide.

[1] The boldface numbers in parentheses refer to the references appended to this note.

The evaporation data may be represented by the form

$$r = A \exp(-b/T),$$

where the values obtained from our data are:

$$A = 2.3 \times 10^9 \text{ g}\cdot\text{cm}^{-2}\cdot\text{sec}^{-1}, \qquad b = 68{,}000^\circ \text{ K}.$$

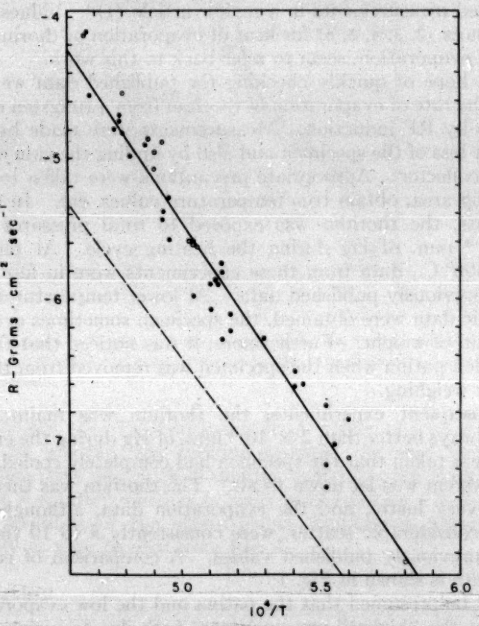

FIG. 1. Specific evaporation rate of thorium vs. reciprocal temperature. Solid points—loss of weight method. Open circles—collector method. Dotted line represents earlier published values (see 6).

The heat of evaporation for thorium as derived from these data is about 140 Kcal·mol^{-1}. With some difficulty, measurements could be carried out on thorium specimens of greater purity and having still cleaner surfaces. In the absence of such measurements, the data presented here are believed to be an improvement on those previously available, and appropriate for many technological purposes.

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BOOK REVIEWS

COSMIC RADIO WAVES, by I. S. Shklovsky, translated by Richard B. Rodman and Carlos M. Varsavsky. 444 pages, illustrations, 6 × 9¼ in. Cambridge, Harvard University Press, 1960. Price, $12.50.

Since the first discovery of radio waves of extraterrestrial origin by Karl Guthe Jansky in December 1931, this new branch of science has made remarkable progress, particularly in the last ten years. It has opened new "windows" enabling us to "peep" through and explore the Universe. The fascinating "views" through these "windows" have been responsible for many discoveries of fundamental cosmological importance that constantly enlarge our concepts about the Universe.

It has now been definitely established that radio waves are emitted from the Sun's gaseous atmosphere from our Milky Way Galaxy, from galaxies in collision at far away distances in space, and from numerous invisible "heavenly" bodies. As the name implies, Dr. I. S. Shklovsky's book *Cosmic Radio Waves* deals with radiation only from objects far away from our solar system. This limitation has resulted in a detailed study of these waves.

The book is divided into six chapters. The first chapter starts with the concepts and definitions of the basic quantities which are measured in radio-astronomical investigations. Then, characteristics of receivers and antennas, the arrangements of antennas to form interferometers, and efforts to increase the resolving power are described. It is stressed at the end of the chapter that the radio-astronomical methods form an essential supplement to optical methods, and real progress depends upon the close cooperation between the two disciplines.

In the second chapter, surveys of the intensity distribution of cosmic radio waves at different frequencies made during most of the major investigations up to the present time are presented. Frequency dependence of the intensity of cosmic radio waves and the spatial distribution of the components of radio emission are discussed. From the analysis of the above data, the important conclusions—that sources of cosmic radio waves form a galactic disc with fairly strong concentration to the center, and other sources of galactic radio waves form a gigantic, almost spherical "halo"—are reached. Numerical data of observations on radio emission from discrete sources their angular extent, and optical identification, are next presented. It is pointed out that the term "radio star," often applied to discrete sources in general, is a misnomer since all discrete radio-wave sources identified up to the present time are associated with nebulae, either galactic or extra-galactic, and since not a single source has been identified with a star. Also it is shown that the spatial distribution of the stronger and weaker discrete sources of cosmic radio waves is opposite to that observed for the stars. The second chapter is full of wealth of the basic observational material, presented in a semi-historical survey, and will serve as a useful ready reference for the research workers in this field.

Chapter 3 deals with problems of the physical processes responsible for observed radio emission from the Galaxy. Arguments that the flat component of galactic radio waves is a consequence of emission by ionized interstellar gas are presented. Evidence for the interstellar magnetic fields comes from the polarization of light from some stars. Emphasis is laid on the conclusion that the magnetic-deceleration theory for the non-equilibrium radio emission of the spherical component can satisfactorily explain all the observational data.

Monochromatic galactic radio waves are discussed in Chapter 4. Observations of the emission and absorption of the 21-cm. line of hydrogen put in the hands of investigators the most powerful tool for studying the spiral structure of the Galaxy, and already extremely valuable information about the physical properties of the interstellar medium, the dynamics and the structure of the Galaxy has been gathered. It is pointed out that the analysis of the 21-cm. line profiles provides

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a new possibility for studying the random motions of clouds of interstellar gas. Possibilities of finding other lines of deuterium, nitrogen and also molecular lines are outlined. This chapter gives a very good account of "radio astrospectroscopy."

The connection between radio-astronomy and the origin of cosmic rays is the subject of Chapter 5. Optical identification of the powerful discrete sources of radio waves with nebulae which are located at the sites of super novae explosions is very suggestive. In order to answer whether all galactic discrete sources of nonequilibrium radio emission be regarded as remnants of super novae explosions, the author carried out an investigation by seeking the information on such explosions in the ancient astronomical records of many countries. He has tried to identify discrete radio sources at such sites, and reached the conclusion that a super nova explodes every 30 to 60 years, on the average. The frequency of such explosions is of extreme importance in questions of cosmic-ray origins. After looking into the nature of the radio emission by the ejecta of super nova, plasma-oscillation mechanism is discarded and the mechanism of deceleration radiation by relativistic electrons in interstellar magnetic fields to explain the peculiarities of the radiation from the ejecta of super nova is accepted. One full section is devoted to the optical continuum and polarization of the crab nebula, which lends support to the deceleration radiation mechanism. The last section of this chapter looks into the question of the origin of cosmic rays. Here the problem is to identify sources in which they arrive, and of acceleration mechanism in these sources. Arguments against the proposd cosmic induction acceleration mechanisms are put forth. Deceleration radiation from the relativistic electrons shows the existences in the interstellar space of particles having energies from 10^8 to 10^9 ev, which is supposed to be the electron component of the primary cosmic rays. The spherical system formed by the sources of the general field of nonequilibrium galactic radio emission outlines the domain of the space within which primary cosmic rays are localized. It is regarded that super nova outbursts maintain the dynamic equilibrium of the cosmic rays in our galaxy. Thus radio-astronomy has opened a completely new approach in our understanding of the nature and the origin of cosmic rays.

The last chapter deals with extragalactic radio waves, and the nature of the radio galaxies. In the end the intimate relation between radio-astronomy and cosmology is stressed.

This book outlines the progress in the field up to the 1958 Paris Symposium on Radio Astronomy. In addition the author has drawn attention to a few selected papers published since that time. In the appendix, these papers are briefly summarized and each one is keyed to the text of the book dealing with the same topic. This has made the book quite up to date. References are grouped under separate years covering the period 1846 to 1960. This chronological order emphasizes the evolution of the field. In addition a general bibliography is provided offering a list of further reading. An Index is provided which includes both authors and subjects.

This book is very useful to the active research workers in the field of radio-astronomy, because it focuses attention upon the major problems in urgent need of further study. Also it should be read by those who are interested in probing the mysteries of our universe.

V. R. POTNIS
Bartol Research Foundation

X-RAY ABSORPTION AND EMISSION IN ANALYTICAL CHEMISTRY, by H. A. Liebhafsky, H. G. Pfeiffer, E. H. Winslow and P. D. Zemany. 357 pages, illustrations, 6 × 9¼ in. New York, John Wiley & Sons, Inc., 1960. Price, $13.50.

This book has been written by a team of authors who themselves have contributed extensively to the recent developments in X-ray emission spectrography (X-ray fluorescence analysis) and X-ray absorptiometry. This is well demonstrated by the author index of 810 entries of which 83 refer to the authors. The book has been written for the analytical chemist or any other worker whose primary interest lies in obtaining an analytical result. This frequently stated intention of the authors has to a great extent influenced the presentation of the subject matter. Any newcomer to the field will find all the necessary information which will enable him to carry out a successful analysis.

In 11 Chapters and 7 Appendices, all well documented by some 450 references, there are treated the origin and properties of X-rays, X-ray absorptiometry with polychromatic and with monochromatic beams, X-ray emission spectrography and the determination of film thickness by X-rays. In the introductory chapters dealing with the physics of X-rays, the presentation follows closely the historical development. Here the aim of the book is most clearly visible; the physical relations are stated, if possible, in a form suitable for numerical evaluation without going into the details of their derivations. The analytical chemist to whom the book is addressed can either take these relations for granted, or he can read the original literature to which frequent reference is made.

In the following chapters, which deal with the spectrochemical analysis with X-rays, an excellent account is given of the methods and tools used at the present time. By frequent reference to practical problems and their solutions, the scope and limitation of every analytical method mentioned are convincingly demonstrated. In order to obtain the same information without the help of this book, many, often obscure, journals would have to be consulted. Here, it seems, lies the importance of this publication. It collects in a well organized manner all the important aspects of the chemical analysis by X-ray techniques excluding X-ray diffraction.

Throughout the text the authors aim at systemizing the terminology which, not surprising for a rapidly developing field, is often confusing. However, the distinction between a spectrometer, which measures the wavelengths and intensities in X-ray beam generated by a tube, and a spectrograph, which measures the same quantities in a beam generated by an unknown sample, appears rather artificial. The term intensity, which plays an important part throughout the book, has apparently resisted consistent interpretation. The statements on page 6 do not give a clear definition of the term intensity, especially when considered in conjunction with the mentioning on page 8 that the integrated intensity can be measured in watts. However, these are minor points which by no means distract from the extreme usefulness of this book.

F. R. L. SCHONING
The Franklin Institute Laboratories

THE THEORY OF LINEAR ECONOMIC MODELS, by David Gale. 330 pages, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1960. Price, $9.50.

This book is an excellent example of what one hopes is a healthy trend in the application of mathematics to the social sciences. Not too many years ago the typical econometrician or psychometrician was an economist or psychologist who utilized in pretty much of a cookbook manner "appropriate" statistical or mathematical techniques with little attention to how they were developed and the assumptions which underlay them. There have been, of course, notable exceptions, as there usually are to such generalizations, and to them we owe the development of these fields as they are today.

Now Professor Gale has written a book in mathematics which is of primary concern to economists, but which few students of economics will be able to read without considerable effort. Mathematicians from other areas will not find it easy either. This situation arises because the modern mathematics which is basic to economic models is just now finding its way into the undergraduate curriculum, and because mathematics is enlarging its role in all the scientific disciplines faster than it is being made a part of most scientists' training. While these remarks may imply a criticism of this volume from one point of view, this reviewer prefers to take the position that "growing pains" cannot be avoided in the development of good quantitatively oriented economists, and that the author's efforts toward this goal are admirable. He sets a standard even though it may be a while before there are many who will meet it.

A brief outline of the contents serves to indicate the scope of the book. Chapter 1 presents examples, definitions, and principal theorems of linear programming. Economic equilibrium which is used throughout as a unifying concept is also introduced here. Chapter 2 developes the needed real linear algebra, from vectors and matrices to convex sets and polytopes, for the remainder of the book. Chapters 3, 4 and 5 discuss, respectively, the theory of linear programming, the simplex method of solution with computations, and the treatment of integral-valued problems such as the optical assignment of

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persons or machines. Chapter 6 is devoted to game theory and covers numerous examples, definitions, and elements of the theory. Chapter 7 takes up the solution of matrix games showing their relationship to linear programming. Chapters 8 and 9 take up the more specialized economic models for exchange and production. The former discusses the simple linear exchange model, dynamic theory, and the place of price equilibrium in exchange models. The latter chapter covers a simple and general linear model, and Neumann models.

The exercises are numerous and varied including proofs, literal problems, and numerical examples. They are effective in promoting understanding at the levels of intended coverage. The typography and format make for good legibility. Assumptions are spelled out so that their effect on the development is apparent.

Two minor points which this reviewer, due to his statistical bias, would like to see included are some mention of stochastic input and its effect on the models, and adherence to what appears to be a helpful convention in books on multivariate analysis—the use of bold-faced type to denote vectors and matrices.

It is hoped that the author will be sufficiently enthused, as non-linear models become more thoroughly investigated, to write a companion volume on them which will be as good as the present text is in its chosen field.

W. JAY MERRILL, JR.
General Electric Co.

CHEMICAL PERIODICITY, by R. T. Sanderson. 330 pages, illustrations, 8 × 10¼ in. New York, Reinhold Publishing Corp., 1960. Price: $9.75 (college); $11.75 (trade).

This is an unusual book, with an unusual approach to chemistry. In his introduction the author wonders "whether civilization in general might not ultimately gain more if scientists would affect a moratorium on all new experimentation for a year or two and spend time in leisurely contemplation of the facts already discovered." It seems that the author has done just about that, focussing his attention primarily on inorganic chemistry. And the conclusions and tables show that his contemplation was far from leisurely.

Modern chemistry is (or ought to be) based on the periodic table of the elements and on the knowledge of the electronic structure of the atoms. The author presents chemistry under this light, and succeeds in showing the internal and logical consistency of the whole field of chemical reactions and affinities. Although our understanding of the chemical bond in molecules and in solids is far from established, it has progressed considerably since the advent of wave mechanics forty years ago. Enough progress has been made to explain not only why atoms combine but also to justify and sometimes even predict many of the properties of the combination (melting points and heat of fusion, boiling points, crystal structure, conductivity etc. . .).

After an introduction to the periodic table (with very pertinent historical remarks) and to chemical combination, the author analyzes the principles of coordination chemistry and the properties of the individual elements. Almost the whole second half of the book is devoted to the binary chemistry of nonmetals. The book contains more than 30 "periodic tables" of the properties of the elements that will prove invaluable. It is probably the first time that the properties of the elements have been presented in such a unified, extensive and logical fashion. The author is to be congratulated for this beautiful synthesis.

It is regrettable that the quantum-mechanical basis of the chemical bond is rather sketchy. It need not have increased seriously the size of the book, since semi-quantitative arguments are possible even in wave mechanics.

With this reservation expressed, this reader feels that this book will contribute enormously to improve the teaching and the understanding of chemistry, and to develop a "more reliable chemical intuition." It extends the concept of electronegativity and proposes several new interpretations. The large format of the book is well suited for the tables, and the publishers should be congratulated for the beautiful and easy to read printing.

HENRI AMAR
The Franklin Institute Laboratories and Temple University

CONFLICT OF INTEREST & FEDERAL SERVICE, a report by the Special Committee on the Federal Conflict of Interest Laws of the Association of the Bar of the City of New York. 336 pages, 6 × 9¼ in. Cambridge, Harvard University Press, 1960. Price, $5.50.

This book touches upon one of the most important problems which faces the Federal Government—"Why is the Federal Government so inept as an employer that the most capable men and women will not undertake or retain Federal employment except in an emergency?" "Why does the Federal Government have such a genius for creating an accumulation of minor annoyances which many of the most able employees are not willing to endure?" These questions have many answers.

Certainly a part of the problem is that the technique for hiring and promoting personnel in Federal employment has been highly unsatisfactory. Thinking in terms of engineers and scientists, Federal employment and promotion has encountered great difficulty in recognizing the person who has learned "things the hard way," but has extraordinary ingenuity, who might be a leading research engineer in a corporate organization. Furthermore, Federal employment and promotion have been very naive about people with paper qualifications who lack real ability.

If Federal employment has encountered serious problems in hiring and promotion, the difficulty in firing an incompetent Federal employee is even more serious. Those who have worked for the Government will remember how incompetent employees are transferred from place to place.

The genius for annoyance of Federal employees is based upon an accumulation of little things. One practice is to give a worker a job to do without providing him with the necessary tools to do it—for example, without permitting him adequate secretarial help, or use of a long distance telephone, or requiring him to travel, and then making a series of unrealistic and inequitable rulings which deprive him of proper reimbursement for travel expenses.

The Association of the Bar of the City of New York is to be congratulated on their recognition of the great importance of this problem and their painstaking, interesting and intelligent consideration of one phase of the problem. They correctly evaluate the present situation:

> The United States has been thrust into a position of world leadership encumbered with the staffing system, traditions, and attitudes toward public service of the nineteenth-century laissez faire government of a second class power. (Page 142)

They recognize that the present morass of confusing statutes regarding conflict of interest is a serious obstacle toward employment of competent personnel by the Federal Government.

They also recognize that this is a particularly important problem to the scientist. According to the report:

> The old caricature of the scientist as a kind of combined Dr. Frankenstein and Mr. Chips has been replaced by a public image of the scientist as a hard working citizen of learning and probity, engaged in vital tasks and worthy of great respect. (Page 167)

Actually the conflict of interest problem from the standpoint of scientific ethics should not be any different for the federally employed scientist than it would be for any other scientist. Any man who is acting as a consultant, or who represents a company doing consulting work is urged to read particularly the illuminating comments regarding the obligation of the scientist in respect to conflict of interest.

The Report realizes that it is completely unrealistic in most cases to impose on an ordinary man entering Federal employment the frightful price exacted of Charles Wilson when he became Secretary of Defense. You will remember that he was required to dispose completely of all of his General Motors stock.

The conclusion reached by this Report is that we should make a completely fresh start in conflict of interest matters of the Federal Government. This is not the place to discuss the merits of the particular statutory changes which are proposed, although they cannot fail to be a great improvement over the existing statutes.

Contrary to the usual rule when a report is written by a committee—"that it takes a committee to understand it"—this Report is beautifully written from start to finish.

A great deal of progress will be made if this book will bring home to those responsible for determining the conditions of Federal employment, the fact that a fresh start is needed not only in the matter of conflict of interest, but on all phases which are concerned with hiring, firing and determining conditions of work of Federal employees.

JOSEPH GRAY JACKSON
Wm. Steell Jackson & Sons

FUNDAMENTALS OF GAS DYNAMICS, (VOLUME III OF HIGH SPEED AERODYNAMICS AND JET PROPULSION), edited by H. W. Emmons. 749 pages, diagrams, 6 × 9 in. Princeton, Princeton University Press, 1958. Price, $20.00.

This book is Volume III of a series of volumes devoted to high speed aerodynamics and jet propulsion. The book opens with a Foreword by Von Kàrmàn acknowledging the support for publication of the series by the Armed Forces.

The contents cover a comprehensive analytical discussion, including basic experimental data on various aspects of gas dynamics. While the separate contributions on subject matter are listed in alphabetical order, by an outstanding group of scientists, they are uniquely edited to form a continuous and excellently coordinated and balanced discussion. Thus the work can be used not only as a reference work but likewise can be introduced as a basic graduate text as a research background for extending our knowledge on gas dynamics.

The opening section, "A," by H. S. Tsien, covers the general equations of gas dynamics.

Section "B" by E. Crow is a one-dimensional treatment of gas dynamics. An excellent discussion of thermodynamic properties including compressibility effects, etc. and covering extensive data on air, precedes the discussion on flow relations. A critical discussion of shock waves, extending to the flow in real nozzles, flow in diffusers, long ducts, is given.

Crow's contributions cover further extensive analysis on various types of flow, in ducts and nozzles, including the effects of heat transfer, specific heats and gas imperfections, of particular value in modern research in various applications of gas dynamics.

Section "C" by A. Kantrowitz is a very interesting discussion on non-steady flow, including the building up of the wave equation, with the effect of entropy changes. The analysis extends to large amplitude isentropic waves. Stability of normal shock waves is followed by a discussion of very intense explosions with a clear physical picture of such phenomena, including examples in rockets, and the "Kadenacy Effect" in internal combustion engines.

Section "D" by Wallace D. Hayes discusses the effect of discontinuities, with its relations to shock wave analysis.

Section "E" by H. Polacheck and R. J. Seeger is on the important subject of shock wave interactions. The work includes a group of unusual photographs of actual effects of shock wave phenomena, such as the regular refraction, air-CO_2 and irregular refraction, air-CH_4 matters, with actual photographs of spinning detonation are shown. Along with this work the authors include an extensive analysis of refraction shocks.

Section "F" by H. Guyford Stever studies the effects of condensation phenomena in high speed flows, and the general kinetics of condensation, including experimental data.

Section "G" by Von Kàrmàn is a basic contribution to general aspects of combustion. In this, he reduces the complicated phenomena of combustion to basic parameters starting with the continuity equation, the momentum equation and the energy equations as constants for the entire process. Then, introducing the concept of diffusion as the mass flux and a parameter for accounting the progress of combustion with chemical kinetics as a function of local temperature, pressure and chemical composition of mixtures, he arrives at generalized equations for the basic phenomena of combustion.

Section "G" is continued with the separate contribution by the editor, H. W. Emmons, on flow discontinuities associated with combustion. This is a scholarly discussion of the complicated subject of flame front extending the fundamental equations of heat transfer, with particular emphasis relating to the boundary conditions and the inadequacy of property reaction rates for the various mixtures. However, the author covers a very illuminating and valuable discussion relating

to flame fronts, both oblique and general and steady flames.

The work concludes with Section G-10 by G. I. Taylor and R. S. Tankin. First the Chapman-Jonquet theory of detonation is discussed, followed by a thorough analysis of general theory of deflagration and detonation, including a very interesting discussion on spinning detonation in gaseous mixtures.

RUPEN EKSERGIAN
The Franklin Institute Laboratories

THEORY OF INERTIAL GUIDANCE, by Connie L. McClure. 340 pages, diagrams, 6 × 9 in. New York, Prentice-Hall, Inc., 1960. Price, $12.00.

There has been a need for some time for an introductory presentation of this popular area of research and especially for relating it to its classical dynamic foundations. This task is undertaken in this book by C. L. McClure who has been an active investigator in this field for many years.

The heart of the subject is dealt with in chapters seven to ten (about two-thirds of the book). The preliminary theory of stabilized platforms is first presented and followed by various single-channel descriptions of Terrestrial Inertial Systems. The presentation is lucid and authoritative. The last chapter on Terrestrial Inertial Guidance Systems and their simulative evaluation is excellent.

In the first six chapters, devoted to a review of the fundamentals of dynamics and gyrodynamics, it is regrettable that a rather clumsy notation has been used making it harder to follow otherwise simple arguments. The appendices and bibliography will be found very useful.

HENRI AMAR
The Franklin Institute Laboratories and Temple University

BIOCHEMISTRY OF STEROIDS, by Erich Heftmann and Erich Mosettig. 231 pages, diagrams, 6 × 9 in. New York, Reinhold Publishing Corp., 1960. Price: $5.75 (College); $6.90 (trade).

The development of the biochemistry of steroids is more recent and has followed the structure elucidation of several classes of steroids by the organic chemists. The publication of "Chemistry of Natural Products Related to Phenanthrene" by the Fiesers in 1936 and its two subsequent expansions, the latest resulting in "Steroids" in 1959, traces the rapid evolution of this field. The present work intends to supplement "Steroids" and describes distribution, biosynthesis, metabolism, physiological significance, structural specificity and analytical determination of each class of steroids.

The classes of steroids are arranged in the sequence of their biogenetic relations. Sterols, steroid sapogenius and alkaloids, cardiac glycosides, bile acids and sex and adrenal hormones are treated in this order. The subject matter is presented in textbook style and in an interesting way. All references (774) are tabulated at the end of the book and are very conveniently classified according to classes of steroids. The illustrations are very clear. Nomenclature and conventions in the representation of structure formulae correspond to those used in "Steroids."

The book should prove very valuable and stimulating for instructional and reference purposes in biochemistry, plant and animal physiology, endocrinology, pharmacology and medicine. Organic chemists also should find the book interesting. Collection of all this material here should help spur further work in this growing and useful field of chemistry.

S. B. MAINTHIA
The Franklin Institute Laboratories

THEORY OF MECHANICAL VIBRATION, by Kin N. Tong. 348 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1960. Price, $9.75.

This book is a welcome addition to the extensive literature on vibration theory. It is essentially a condensed theory on vibration analysis, extending from single-degree-of-freedom systems to multi-degree-of-freedom systems. Special emphasis is given to matrix algebra extended to generalized coordinates of elastic bodies and continuous media. An interesting discussion on theory and application of eigenfunctions is included.

The examples are well chosen, covering important applications to transducers, lateral and torsional vibration of shafting, etc. In the analysis of a rolling load on a bridge beam, the author analyzes correctly the effect

of the inertia of the moving load in addition to its weight, using the interaction of the moving load as acting on the eigenfunction coordinates of the beam considered separately, though the author notes the equations are not readily solvable.

One valuable feature of this book is the appendix on the matrix algebra in linear transformation of vectors, which use in turn is a characteristic feature of this book.

The author is to be complimented on a remarkably complete and condensed theory of vibration analysis, covering a range of interesting topics.

R. EKSERGIAN
The Franklin Institute Laboratories

PHYSICS AND MEDICINE OF THE ATMOSPHERE AND SPACE, edited by Otis O. Benson, Jr. and Hubertus Strughold. 645 pages, illustrations, 6 × 9 in. New York, John Wiley & Sons, Inc., 1960. Price, $12.50.

In November of 1951 a symposium on the "Physics and Medicine of the Upper Atmosphere" was held at San Antonio, Texas. It was sponsored by the Air University School of Aviation Medicine. About a year later, the proceedings of this symposium was published with one of the editors, Brig. General Otis O. Benson, Jr., Commandant of the USAF School of Aviation Medicine.

Seven years later in November 1958 the second symposium was held in San Antonio and the proceedings of this meeting has been published as *Physics and Medicine of the Atmosphere and Space*. Again, one of the editors is General Otis O. Benson, still heading up the USAF School of Aviation Medicine.

With this type of editorship, we should expect this new book to parallel the impact of the first symposium proceedings. In truth, that is exactly what has taken place. The Arrangements Committee has once more invited the ranking scholars in the many disciplines which border space flight. In the 1951 symposium only one man was invited from outside the United States. The 1958 symposium saw five authorities from the free world present contributions.

The symposium proceedings covered all facets of space flight. Some this reviewer considers as extraordinarily comprehensive and complete. As an example, Dr. S. Fred Singer covers "Interplanetary Dust and Radiation Environment." Because of the reviewer's research activities he was particularly interested in this section which is magnificently treated. Dr. Leslie R. Shepherd contributes a "Survey of Possible Propulsion Systems" which again is excellent exposition. With characteristic thoroughness he covers these systems from the chemical to the photon rocket. The indefatigable Krafft A. Ehricke's contribution on "Manned Orbital and Lunar Space Vehicles" represents a rather complete wrap-up of man's steps into space to ultimately land on the moon. This contributor again stresses the need for the use of nuclear power plants in these space missions but it should be remembered that this contributor has for a long time advocated this approach.

With the astronomical background of this reviewer it is natural for him to give particular attention to the papers by Dr. Roberts on the Sun; Dr. Kuiper on the lunar and planetary environments and Dr. de Vancouleurs on Mars.

The pace of astronautics is moving forward so rapidly that it leaves even the experts in its wake. As examples of this progress, Dr. Van Allen, in his article on radiation hazards, speaks of space flight at relativistic velocities and shows that the radiation exposure level reaches 20,000,000 roentgens per hour. He also indicates the erosive effect of particles on the space ship at this speed. This reviewer has considered this problem in a recent book and concludes that the only solution to the problem of relativistic velocities is the use of an asteroid with walls several hundred feet thick.

Also we find in this book in two places a perpetuation of a fallacy. Drs. Gerathewohl and Ward in their "Psychophysiologic and Medical Studies of Weightlessness" and Dr. Stapp in his "Human Tolerance to Accelerations of Space Flight" indicate that Laika was weightless in Sputnik II.

This reviewer has studied this problem in light of the rotation of the capsule at about once every four seconds and assuming the vestibular organ of the dog was 5 feet from the center of rotation. He finds that the centrifugal force to which Laika was subject is the equivalent of a $\frac{1}{4}$ g field! Thus, unless the head of Laika was precisely at the center of rotation she was not weightless and the

physiologic responses were not compatible with the absence of a g field.

On the whole, this companion volume to "Physics and Medicine of the Upper Atmosphere" is a most welcome addition for the serious student and worker in the space flight sciences. The Arrangements Committee and Editors are to be congratulated on the uniform excellence of this volume.

I. M. LEVITT
The Fels Planetarium
of The Franklin Institute

CLASSICAL DYNAMICS, by R. H. Atkin. 273 pages, diagrams, 5½ × 8½ in. New York, John Wiley & Sons, Inc., 1959. Price, $5.25.

Classical dynamics and electrodynamics constitute the essential basis of classical physics. Yet the relative time alloted them in academic curricula has been progressively reduced, in view of the rapid growth of many modern branches of physics (atomic, nuclear and solid state physics, for example). Hence the need for well written and concise presentations. To the excellent treatises of Goldstein, Synge and Griffith, etc., Atkin has added a new one which, in a sense, is more old fashioned and more elementary.

The vector notation is used throughout, and the mathematical preliminaries are introduced in the first two chapters. The central force problem is presented with clarity and completeness, and so are the topics of rigid body dynamics, and normal coordinates. The examples solved are well chosen and illustrate directly the principles. The numerous problems proposed at the end of the chapters, taken mostly from recent English examinations, are rather traditional and narrow in scope. They will nevertheless contribute to give a good working knowledge to the students.

It is however regrettable that some topics of considerable importance in modern physics have been omitted. Let us mention the scattering problem, the Hamilton-Jacobi Theory and relativistic dynamics among others. The rich illustrations provided by electrodynamics have been neglected. Regrettable also and rather surprising is the lack of an index.

Despite these omissions, which can be remedied by an instructor, the topics considered are treated with an unusual skill. The working knowledge has been emphasized at the expense of the philosophical foundations. As it stands this little book is eminently teachable and could constitute an excellent text at the junior-senior level.

HENRI AMAR
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BOOK NOTES

THEORY OF THERMAL STRESSES, by Bruno A. Boley and Jerome H. Weiner. 586 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1960. Price, $15.50.

This volume gives a comprehensive account of all the presently available theoretical techniques for determination of stresses produced in a solid object when it is subjected to specified heating conditions. Thermal stress is explained through the basic subjects of thermodynamics, heat-transfer theory, elasticity and inelasticity. Realizing that many persons involved in thermal-stress analysis have little familiarity with heat-transfer theory, a self-contained review of the subject—together with a discussion of methods of problem formulation and methods of their solution—has been included. In addition to the presentation of the theoretical foundations, methods of practical thermal-stress analysis, utilizing strength of materials approximations, are treated fully in both the elastic and inelastic domains.

SELF-SATURATING MAGNETIC AMPLIFIERS, by Gordon E. Lynn, Thaddeus J. Pula, John F. Ringelman and Frederick G. Timmel. 217 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1960. Price, $8.00.

This thorough treatise on self-saturating magnetic amplifiers, on facts of their operation and on precise accurate design techniques, shows the reader how to design complex magnetic amplifier circuits with a minimum of cut-and-try experimentation. Clearly ex-

plained in line with readily observable laboratory phenomena is the true operation of a physically realizable amplifier, and idealizations of nonlinear functions are made only after the real conditions are discussed. Among the subjects covered are the basic assumptions involving circuital laws, modern techniques for dealing with the transient behavior of self-saturating magnetic amplifiers, transformer theory, feedback control theory, and magnetic theory. In addition, there are facts on the design of magnetic amplifiers, testing methods applicable to magnetic core material, and a number of commonly encountered design problems.

TRANSACTIONS OF THE VACUUM METALLURGY CONFERENCE 1959, Edited by Rointan F. Bunshah. 212 pages, illustrations, 8½ × 11 in. New York, Associated College Presses, 1960. Price, $7.50 (paper).

This volume is a collection of the papers presented at the third annual meeting on vacuum metallurgy held at the University Heights campus of New York University, June 1–3, 1959, under the sponsorship of the Department of Metallurgical Engineering in co-operation with the Office of Special Services to Business and Industry. This conference, at which papers on new research and advanced engineering information in vacuum metallurgy were presented, followed in logical succession the two previous meetings which were courses consisting of a planned series of lectures designed to familiarize the engineer with the vast and diverse areas embraced by the term "vacuum metallurgy."

THE CELL, by Carl P. Swanson. 114 pages, illustrations, 6 × 9 in. New York, Prentice-Hall, 1960. Price: $2.95 (cloth); $1.50 (paper).

First in the Foundations of Modern Biology Series, this text opens up the whole fascinating world of cellular phenomena and gives the reader an idea of the place of the cell as the basic unit of biological structure and function. The cell concept is discussed as it was originally proposed and as it is now interpreted. It is considered in terms of its origin through cell division, correlates structure and function, and traces the cell through the processes of growth, differentiation, reproduction and death. *The Cell*, along with its ten companion volumes in the series soon to be published, constitutes a new and vital contribution to general biology literature. This clearly written and well illustrated book gives an outstanding treatment of its own subject area, and at the same time, is an integral part of the whole series.

ANIMAL GROWTH AND DEVELOPMENT, by Maurice Sussman. 114 pages, illustrations, 6 × 9 in. New York, Prentice-Hall, Inc., 1960. Price: $2.95 (cloth); $1.50 (paper).

This second offering in the Foundations of Modern Biology Series reflects the latest research findings and discoveries based on present-day concepts of cell physiology and genetics. Contrasted with the usual classical treatments, the subject matter is not restricted merely to embryonic development. The book begins with a consideration of single celled organisms and of phenomena associated with the development of a cell from its appearance as a daughter cell to its disappearance as a mother. The life cycles of typical microorganisms are also described as well as sexual phenomena, morphogenetic processes, and persistent rhythmic developments are discussed. Readable, authoritative and accurate, this valuable study refers throughout to the major unsolved problems of 20th century biology and points out the exciting work which remains to be accomplished by research scientists of the future.

THE CORRESPONDENCE OF ISAAC NEWTON. VOLUME II, 1676–1687, edited by H. W. Turnbull. 552 pages, illustrations, 8 × 11 in. New York, Cambridge University Press, 1960. Price, $25.00.

Volume II of this series sponsored by the Royal Society covers the years 1676 to 1687 and includes, as in Volume I, correspondence not only of Newton himself, but also of others (such as Collins, Oldenburg, Leibniz, Hooke, Halley) whose letters to each other bear a relation to Newton's own letters and other writings. The relationships among the letters are carefully noted, so that it becomes easy to trace through a series of letters, both prior to and following any given piece of correspondence. Those letters which were written in German or Latin are printed in their orig-

inal text, followed by English translations. The eleven-year period covered by Volume II is of particular interest, since it is climaxed by the publication of the *Principia* in 1687, and several of the items (notably by Halley and Newton) deal with events leading up to that publication. Volume II meets the excellent standards set in the first volume, reviewed in this JOURNAL for February, 1960.

OXIDATION–REDUCTION POTENTIALS OF ORGANIC SYSTEMS, by W. Mansfield Clark. 584 pages, diagrams, 6 × 9 in. Baltimore, The Williams & Wilkins Co., 1960. Price, $13.50.

Herein are presented more than 100 tables summarizing existing data on groups of organic oxidation-reduction systems. Along with critical notes and a bibliography, these tables form a valuable reference work. The remainder of the text (over half the book) is divided into thirteen chapters dealing with techniques, methods, apparatus, theories and definitions of many phases of organic chemistry. The author points out restrictions of a theoretical nature, as well as limitations of particular experimental methods.

MY LIFE AND TIMES AS A PHYSICIST, by I. I. Rabi. 55 pages, 5¼ × 8¼ in. Pasadena, The Castle Press, 1960. Price, $2.75.

Two lectures by I. I. Rabi provide thoughtful readers an opportunity to take stock of the position of science in community life, now and for the future. The first lecture, "The Scientist and the Statesman," makes a sound plea for a more adequate understanding of science among the leaders of our national government. The second lecture on "Science and the Humanities at Mid-Century" stresses the importance of having nonscientists understand the scientific point of view (particularly students). Dr. Rabi develops his ideas by means of examples, both historical and autobiograqhical, couched in lucid, plainly understandable language. His purpose is to ". . . find ways to bring a greater unity to our culture . . ."; he believes that "The appeal of science lies very deep in men, because it expresses fundamentally the desire to explore, to understand, to know how things hang together, . . . and not least, to utilize this knowledge for human purpose." His conclusion is that "Only by the fusion of science and the humanities can we hope to reach the wisdom appropriate to our day and generation."

DIGITAL COMPUTER AND CONTROL ENGINEERING, by Robert Steven Ledley. 835 pages, illustrations, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1960. Price, $14.50.

Intended as an elementary text for seniors or first-year graduate students in engineering, this book deals with all aspects of digital computers from an engineering point of view. However, the book will prove useful in industry as well as in the classroom. Three major phases are treated: (1) the over-all design of digital systems; (2) the logical design of digital circuitry; and (3) the electronic design of digital circuits. Among the topics covered thoroughly are programming and coding, numerical analysis, Boolean algebra, computational methods, matrices, arithmetic operations, semiconductor and magnetic elements, etc. An appendix contains several useful tables of formulas.

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PUBLICATIONS RECEIVED

PUBLIC RELATIONS AND MANAGEMENT, by David Finn. 175 pages, 5 × 7½ in. New York, Reinhold Publishing Corp., 1960. Price, $4.50.

CAREERS IN PHYSICS, by Alpheus W. Smith and Winston L. Hole. Revised edition, 310 pages, illustrations, 6 × 9 in. Columbus, Long's College Book Shop, 1960. Price, $5.95.

NEW DESCRIPTION OF THORIUM SPECTRA, by Romuald Zalubas. National Bureau of Standards Monograph 17, 103 pages, 8 × 10½ in. Washington, U. S. Government Printing Office, 1960. Price, $0.65 (paper).

SCIENTIFIC BASIS FOR WORLD CIVILIZATION, by Leo J. Baranski. 649 pages, 6 × 9½ in, Boston, Christopher Publishing House, 1960. Price, $7.50.

THE SKELETON KEY OF MATHEMATICS, by D. E. Littlewood. Reprinted from the first edition (1949), 138 pages, diagrams, 5½ × 8 in. New York, Harper & Brothers, 1960. Harper Torchbook 525. Price $1.25 (paper).

THE HIGHER ARITHMETIC, by H. Davenport. Reprinted from the 1952 edition, 172 pages, diagrams, 5½ × 8 in. New York, Harper & Brothers, 1960. Harper Torchbook 526. Price, $1.35 (paper).

A SHORT HISTORY OF CHEMISTRY, by J. R. Partington. Reprinted from the third edition (1957), 415 pages, illustrations, 5½ × 8 in. New York, Harper & Brothers, 1960. Harper Torchbook 522. Price, $1.95 (paper).

A DIRECT ENTRY TO ORGANIC CHEMISTRY, by John Read. Reprinted from revised (1959) edition, 268 pages, diagrams, 5½ × 8 in. New York, Harper & Brothers, 1960. Harper Torchbook 523. Price, $1.50 (paper).

CHEMISTRY IN THE SERVICE OF MAN, by Alexander Findlay. Reprinted from the eighth (1957) edition, 318 pages, diagrams, 5½ × 8 in. New York, Harper & Brothers, 1960. Harper Torchbook 524. Price, $1.75 (paper).

A STRUCTURE OF SCIENCE, by Joseph H. Simons. 269 pages, 5½ × 8½ in. New York, Philosophical Library, 1960. Price, $4.75.

ENERGIES OF THE UNIVERSE, by Eugene Fritz. 124 pages, diagrams, 5½ × 8½ in. New York, Philosophical Library, 1960. Price, $4.75.

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INTERCOMPARISON OF LABORATORY TEST METHODS

A method for evaluating mathematically the factors involved in the variability of interlaboratory test results has been developed by J. Mandel and T. W. Lashof of the National Bureau of Standards.[1,2] This "model" for statistical analysis has been successfully applied to several problems involving both physical and chemical properties of materials. A better appraisal of test methods can be made by utilizing the method to distinguish between random and systematic sources of error.

While the model is designed primarily for the study of a single test method on an interlaboratory scale, it can also be used for the interlaboratory comparison of two alternative test methods. For this purpose, it is applied in conjunction with another statistical procedure, the sensitivity criterion,[3] involving the functional relation between the results obtained by use of the two alternative test methods.[4]

The true reproducibility and therefore the usefulness of a method of measurement are unknown until the method has been evaluated on an interlaboratory scale. However, the interpretation of data from interlaboratory round robins is not always straightforward. There has been need for a clearly formulated model that would relate mathematically the various factors entering into the measurements. In recent years, considerable attention has been given to studies of test methods involving relatively large numbers of laboratories. These studies resulted from the observation that even when well-established methods of measurement are used, the agreement among laboratories is, in general, much less satisfactory than the agreement which can be achieved within a single laboratory.

The sources of variability usually considered in such studies include:

1. Replication error, or differences among results obtained at one laboratory by one operator using the same instrument on specimens of the same material;

2. laboratory main effect, or systematic constant differences among results obtained at different laboratories; and

3. laboratory-material interaction, or variations among laboratories in the differences among results obtained for various materials.

The present method separates the third source of variability into two components; one is random, and the other reflects systematic laboratory differences

[1] "The Measuring Process," by John Mandel, *Technometrics*, Vol. 1, p. 251 (1959).

[2] "The Interlaboratory Evaluation of Testing Methods," by John Mandel and T. W. Lashof, *ASTM Bul.*, No. 239, p. 53 (1959).

[3] The sensitivity of a method A used in the study of a property Q is defined as $(\Delta A/\Delta Q)/\sigma_A$, where A is the measurement obtained and σ_A is the standard deviation of method A.

[4] "Evaluation of Test Methods by the Sensitivity Criterion," *NBS Tech. News Bul.*, Vol. 40, p. 139 (1956).

that cannot be represented by constant biases. To account for the random scatter one must consider, in addition to replication error, the effects, upon the measured value, of properties other than that property being measured. The interfering effect of such properties often varies among laboratories, causing additional scatter in the interlaboratory results. Another source of random scatter is improper calibration of equipment. In certain instruments having multiple scales, part of the random scatter has been shown to arise from lack of continuity among the scales or from variation in the ratios among the scales.

In the absence of systematic errors, a graph of the results from one laboratory plotted against those from another shows a scatter of data points about a straight line bisecting the angle formed by the axes (a line of 45° slope). If the results from one laboratory were consistently higher or lower by a constant amount than those from the other, the line would be higher or lower but would be parallel to the first line; that is, it would lie at 45° to either axis but would not pass through the origin.

In practice, the lines obtained are not of 45° slope, and it is this nonparallelism or slope effect that reflects a varying systematic difference between the two laboratories. For example, an error in the determination of the normality of a reagent will cause a proportional type of error in all analyses employing this reagent. For any number of laboratories, the slope effect can be generalized by plotting the results from each laboratory against the average of the results from all.

The separation of various error sources provided by this method permits evaluation of possible benefits derivable from reference samples. In the case of constant differences among laboratories, a single reference sample is sufficient. If the slopes of the lines plotted for various laboratories differ to a great extent, the use of two reference samples is indicated. Suppose that specimens of two reference materials are supplied to all participating laboratories. The two materials, although necessarily similar in nature, must differ considerably in the value of the property under test. The values for this property obtained by measurement in any given laboratory for these two materials can be compared with the known correct (or assigned) values; then at each laboratory the measured values for any submitted unknown sample of such material can be "adjusted" and brought into line with the similarly adjusted values from other laboratories.

The use of reference samples to improve interlaboratory precision is an accepted procedure and differs little from some calibration methods; however, reference samples should be used for this purpose only after all economically feasible improvements in the control of environmental conditions or other desirable modifications of the test procedure have been exhausted.

Most statistical methods are highly sensitive to extremely discrepant values. The present method contains several safeguards against misleading results due to the inclusion of outlying measurements. Standard deviation of replication error is examined for each material and laboratory. Single discrepant values are then easily detected from inflation of the corresponding standard deviations. Trends are handled statistically through appropriate transformations of scale. If all the replicates obtained by one laboratory for a particular material are affected by the same error, the contribution of this laboratory to the random part of the laboratory-material interaction will be

unduly large and consequently obvious. A laboratory discrepant in all values will be apparent from the fact that its line will differ from the others appreciably in position or slope.

Sometimes these discrepancies can be clarified and the nonconforming values replaced by the results of measurements taken under proper conditions. When no reason for the conflict can be found, the values must be considered as possibilities in the application of the test method.

A CARBON-14 STANDARD FOR LIQUID-SCINTILLATION COUNTERS

The National Bureau of Standards has prepared a new carbon-14 standard sample to meet the need created by the increased use of liquid-scintillation counters, particularly in biochemical and medical studies. The solvents most often used in these counters are toluene and xylene; the standard used in their calibration must therefore be soluble in and compatible with these compounds. To fulfill this requirement, W. F. Marlow and R. W. Medlock of the Bureau's radioactivity laboratory used benzoic acid-7-C^{14} dissolved in toluene for the new standard.[5]

In the past few years, liquid-scintillation counting systems have become increasingly accurate and reliable, while their cost has greatly decreased. The sample of material to be counted by this method is more easily prepared than for most other methods, and since it is dissolved directly in the solution, the counting efficiency is usually much greater, especially for low-energy radiations of carbon-14.

The sample of benzoic acid-7-C^{14} in toluene was standardized by quantitative oxidation of the benzoic acid and toluene. The carbon dioxide produced was collected and the level of radioactivity of this carbon dioxide was compared, in an ionization chamber, with the radioactivity of carbon dioxide prepared quantitatively from the national sodium carbonate-C^{14} standard.

Previous work[6] has shown that $C^{14}O_2$ and $C^{12}O_2$ may be evolved at different rates during oxidation of organic compounds either by wet combustion or by burning in an oxygen stream. The possibility of error due to this isotope effect was avoided by burning the sample in a modified Paar oxygen bomb to obtain complete oxidation.

The new carbon-14 standard sample may be obtained from the Radioactivity Section, National Bureau of Standards, Washington 25, D. C. for $27.00.[7]

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[5] For further information, see "A Carbon-14 Beta-Ray Standard, Benzoic Acid-7-C^{14} in Toluene, for Liquid-Scintillation Counters," by W. F. Marlow and R. W. Medlock, *J. Research NBS*, Vol. 64A, p. 143 (1960).

[6] "Errors of Combustion of Compounds for C^{14} Analysis," by W. D. Armstrong, L. Singer, S. H. Zbarsky and B. Dunshee, *Science*, Vol. 112, p. 531 (1950).

[7] A complete listing of standard samples is contained in Standard Materials, NBS Circular 552 (third edition), which may be obtained by writing to the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C. (35 cents).

CURRENT TOPICS

Satellites Inflated with Plastic Foam?—Tomorrow's space stations and special-purpose satellites may be built with balloon-like tubes that can be folded into compact packages for launching and inflated automatically in space with a plastic foam hardened by the sun's heat, according to an engineer of the Radio Corporation of America.

This technique is now under study at RCA's Astro-Electronics Division, Princeton, N. J., as a possible solution to the problem of setting up in space very large structures, such as extensive solar energy collectors and radar antennas 60 ft. or more in diameter. It was described by Carl C. Osgood, Manager, Design Engineering, of the RCA division, in a paper prepared for presentation at the annual West Coast meeting of the American Astronautical Society.

Mr. Osgood said the use of inflatable tubes promises to overcome the size and weight limitations encountered in launching folded structures designed to be set up in space. As an example, he pointed out that an inflatable tube structure weighing only 32 lb. could provide a disk 44 ft. in diameter capable of carrying a solar cell array weighing 420 lb.

"For smaller short-life applications, a gas bottle large enough to inflate and maintain the leakage rate of the array is considered satisfactory," said Mr. Osgood. "But for the larger, longer-life devices, the maintenance of shape and rigidity by continuous gas pressure is inadmissible because of the weight of tankage. An arrangement of rigid ribs and hingelock combinations appears feasible, but is eliminated because of the high weight of the many pieces of hardware and frequent difficulty in meeting the folding requirements.

"Additional search for methods of inflation leads to the use of 'foamed-in-place' structural members. This process makes use of the increase in volume—from three to fifty times between liquid reactants and the solid foam product. Before curing, the foam has some fluid properties, and its pressure, plus that of the escaping by-products, can be used as the inflation pressure. The cured foam could provide the permanent rigidity required to retain the shape of the array. This would obviate the need for continuous gas pressure or for transporting rigid structural members."

Mr. Osgood said that engineers at RCA's Princeton Space Center are now investigating methods of foam inflation and the development of plastic foam materials sufficiently resistant to the radiation encountered in space. He discussed several possible methods of foam inflation based on combination of the reacting elements by their simultaneous injection into the inflatable structure when the satellite is released in orbit from the launching vehicle.

A further method under study, according to Mr. Osgood, is the use of "frozen" materials that would be combined in the collapsed satellite structure before launch and kept below the reaction point by refrigeration techniques until the orbit is achieved. In orbit, solar heat would start the reaction, generating the foam to expand the structure, and curing the foam to the desired hardness.

Polaris Missile Trainer.—The first production-model Polaris Fire Control trainer has been delivered to the United States Navy by General Electric Company. Designed to train crews aboard fleet ballistic missile submarines, the electronic device permits simulated submerged launchings of the intermediate-range Polaris.

It is scheduled for installation aboard the nuclear-powered submarine *Theodore Roosevelt*, now under construction at Mare Island Navy Yard, Calif. Such trainers will be standard equipment aboard *George Washington* (SSB(N)598) and *Ethan Allen* (SSB-(N)608) class fleet ballistic missile submarines, according to the Navy.

The computer-like trainer can be connected to the fire-control system of the submarine, allowing its attack party to practice complete launching operation on the same equipment that would be used to fire an actual missile.

As the crew operates the fire-control system, the device is so versatile that it creates the conditions of an actual missile launching. It can even simulate a wide variety of malfunction for the attack party to cope with.

General Electric conducted a full-scale team effort among several of its departments to design and produce the training device for the Naval Training Devices Center, Port Washington, N. Y.

Its General Engineering Laboratory, Schenectady, N. Y., developed the complex electrical design at the direction of the G. E. Ordnance Department, Pittsfield, Mass.

The company's Specialty Control Department, Waynesboro, Va., was selected as fabricator of the equipment because of its wide range of manufacturing capabilities and techniques and its experience in building similar special and defense control equipment.

Contracts for production-model trainers have been let to the G. E. Ordnance Department and will be manufactured by the Specialty Control Department.

Subminiature Accelerometer.—The development of a new subminiature piezoelectric accelerometer, Model 606, designed for use in special shock and vibration test applications where small size and light weight are the prime considerations, has been announced by Columbia Research Laboratories, Woodlyne, Pa.

The model 606 is a stainless steel accelerometer only $\frac{1}{4}$ in. high, weighing just 1.0 gram, permitting many in-flight tests that could not normally have been conducted with standard size or miniature size pickups. The unit has a sensitivity of 2.5 pk-mv/pk-g and a frequency response of 5 cps to 10 kc with its resonant frequency at 135 kc. Maximum acceleration is given at 40,000 g, amplitude linearity $\pm 2\%$, and temperature range $-65°$ F. to $+300°$ F. for standard units and $-65°$ F. to $+500°$ F. for high temperature units, with less than ± 10 per cent variation in sensitivity throughout both temperature ranges.

Satellite Observatories.—A project to launch satellite observatories capable of taking very accurate astronomical measurements from high altitudes in space is under way at the University of Wisconsin, one of five institutions participating in NASA's orbiting observatories program. The others are Harvard, Princeton, the Smithsonian Institution, and NASA's own Goddard Space Laboratories. For the immediate future, in the cooperative effort, work will be concentrated on launching three satellites to carry telescopes of various sizes and to be devoted to problems concerned with ultra-violet light.

At altitudes in excess of 100 miles, the "flying" observatories will escape the problems presented by the Earth's atmosphere. They should be able to see the ultra-violet light that comes from the stars and to transmit their measurements to earth by electronic methods. The telescopes should be able to point at a star with more accuracy and precision than earthbound ones.

The first of the orbiting observatories will carry a number of small telescopes for UW experiments on the brightness of stars in the ultra-violet. Wisconsin will share this satellite with the Smithsonian Institution which will be doing photography of the skies by television techniques in the ultra-violet. The second satellite will carry an ultra-violet spectrograph for the Goddard Space Labs; the third, a high dispersion ultra-violet spectrograph for Princeton.

It is planned to lift the observatories into orbit with Atlas rockets topped with Agena B boosters. A number of these Agenas have already been successfully launched in other space experimentation. Wisconsin hopes to get the first of the orbiting observatories into space by 1963. UW scientists hope, also, that once orbited, a satellite will continue to operate satisfactorily for at least a year.

UW astronomers will be responsible for the design of the telescope instrumentation and for everything necessary to carry out the actual experiments. They will construct experimentally some parts in order to test out certain ideas.

Though the first satellite will carry relatively small telescopes for Wisconsin investigators, UW astronomers expect to launch later in the program a single telescope as much as 36-in. in diameter and something in the order of a ton in weight.

Ever since the beginning of astronomy, scientists have had to observe from the bottom of a "sea" of air and this atmosphere, more than any other thing, has limited what can be learned about the nature of the universe. It bends the light, making the images of the stars being measured fuzzy and seeming to move. Measurements in the finest detail are not possible. The blanket of air also scatters the light, thus producing a sky brighter than it would be otherwise. Because of this, the faintest objects cannot be seen. In addition, the atmosphere presents the problem of being opaque to most electro-magnetic radiation.

It is possible to escape the bending and scattering of light by making observations from balloons, but only by reaching altitudes in excess of 100 miles (by means of the flying observatories) can astronomers see the ultra-violet that comes from the stars. So, the first programs in space astronomy will be devoted to measuring ultra-violet and radio light.

Possibly within the next year, NASA will precede the actual launching of the orbiting observatories with astronomical measurements taken in relatively short periods of observation from rockets or from an X-15, manned rocket plane. By the fall of 1961, the X-15 is expected to reach the heights necessary. Whatever the vehicles, rockets or planes, most of them will carry photo-electric equipment and the records they gather will be radioed to Earth.

All of the institutions concerned with the orbiting observatories will ultimately share the use of the "flying" telescopes, and astronomers not associated with the project will also have opportunities to make observations with the satellites.

Test-Pilot Monitor.—A new manned-vehicle monitor that can tell at a glance the immediate condition of a pilot and craft he is flying has been designed for the Air Research and Development Command (ARDC) by International Telephone and Telegraph Laboratories, San Fernando Valley, Calif.

The new device, to be operated by ground personnel, for the first time will give direct read-out or "real time" safety reports on the pilot's physiological, and environmental condition. In addition, "how goes it" information on the vehicle itself is transmitted to the ground observers.

Designed and in development under contract for ARDC's Air Force Flight Test Center, Edwards AFB, Calif., the space-data display unit shows the changing environmental and physiological conditions of a man before he may be aware of them. Safety control from the monitoring point will be instantaneous by communication to the vehicle.

According to Paul Sehnert, AFFTC, Project Engineer, and Carlos Miller, Executive Engineer of ITTL, the "real time" console will keep abreast of man's conditions within fractions of a second and can indicate immediately when he is undergoing or entering an unsafe condition. No computation of data, graph, or gauge analysis is necessary, they said.

"We need to know immediately the status of a man's condition," Sehnert said, "and with this new device complex information is shown and read without need of lengthy and perhaps very costly analysis."

The development work currently under way follows a feasibility study. The system is specifically being built to monitor experimental high-performance aircraft or rocket plane flights such as the X-15. It may be used on other space-type vehicles or man-in-space programs.

The electronic design of the new system follows the most advanced state-of-the-art. Solid state components are used wherever practicable, and miniaturization and reliability are exercised throughout the console.

The first unit will be ready for installation and operation early next year at Edwards AFB.

The console is separated into two units, vehicular and physiological, each about half the size of an office desk. A flight test engineer will monitor the vehicular unit while a flight surgeon will monitor the physiological unit, but they will only indirectly, by voice communication, exercise control of the vehicle through advice to the pilot.

The information display on the vehicular panel includes various stability and aerodynamic parameters, structural strains, flutter, power plant behavior, and others such as skin temperature, fuel, fire warning, electrical equipment, etc.

The information shown on the physiological panel includes the man's heart rate, blood pressure, temperature, breathing rate and volume, the parameters concerning his environment, oxygen supply, acceleration, radiation count, etc.

Four basic kinds of display media are used to report the so-called "how goes it" information: Cathode ray tubes generate, for two inter-related parameters, a spot of light on an X-Y plot, and for a single parameter, a marker line on a thermometer-type display; indicator lights show "safe" or "unsafe" conditions of certain parameters; counter wheels show quantitative value of selected parameters; and time-history displays, for certain physiological parameters, show recent history to allow accurate interpretation of the real-time value.

The cathode ray displays will present a green color indication of the "OK" condition which changes to a harsh orange when either the actual value or the rate-of-change becomes unsafe. For the X-Y plot displays of inter-related parameters, the green spots of light are located inside "safety envelope" areas which have been previously determined for the particular parameters for both the man and the vehicle. The spots of light change their color to orange when the parameters move outside the "safety envelope" areas.

Rotating Microscope Stage.—Said to have almost unlimited possibilities in laboratory and production-line work, a brand-new Rotating Microscope Stage now permits microscopic top-and-side examination of tiny subjects without moving them or holding by hand. Developed by Edmund Scientific Company for use with low power microscopes, the new Rotating Stage is now finding use for the microscopic inspection and study of tiny components and assemblies, material samples, contact lenses, resistors, transistors, miniature gears, gems, minerals, biological and other specimens. Subject is placed on the revolving stage for study under the scope. A touch of the finger revolves the center platform and rotates the subject for examination. An angled first surface mirror permits top-and-side viewing without the necessity of "upsetting" the subject or awkward holding by hand. Flat edge of rotating wheel indicates complete revolution. Unit is sturdy metal with plastic rotating parts—size 2 in. × 2 in. × 2 in.

World's Fastest Switching Device.—The world's fastest switching device—a radically improved tunnel diode that could conceivably count ten billion one-dollar bills in a second or make computer "decisions" with equal speed—has been introduced by the Radio Corporation of America.

Already, the Department of Defense is studying these new tunnel diodes with a view to developing a computer that will operate at a speed of a billion cycles per second—100 times faster than any computer in existence, according to Kenneth M. McLaughlin, Manager, Computer Products, RCA Semiconductor and Materials Division.

"The device is one of four new types already in production and available commercially," he said. "Because of their tremendous speed, they are expected to have wide use in a variety of electronic systems including data processing, missiles, satellites and various communications equipment."

Tunnel diodes—tiny solid-state devices no bigger than the head of a match—control the flow of electrons that make up an electrical current much faster than an electron tube or a transistor. They are called tunnel diodes because of the manner in which electrons seem to "tunnel" through them with the speed of light.

"These new diodes operate at incredible speeds," Mr. McLaughlin said. "One of them is fifty times faster than the fastest transistor. As an illustration of their speed, Mr. McLaughlin said the fastest of the new diodes can switch from one voltage to another in the time it takes light, traveling at 186,000 miles per second, to traverse a distance of about one inch.

"Theoretically, then, it is possible for this device to switch—or, in popular terminology, 'to make decisions'—ten billion times a second."

The low prices listed for the new diodes are a result of both technological advances that have reduced production costs and a growing market resulting from an awakening to the

tremendous versatility, reliability and promise of these rugged and very tiny components.

Repair Stations for Spacecraft.—Emergency drydocks and "space hogs" have been proposed for fixing spacecraft "flat tires." The problems of fixing an ailing spacecraft and possible solutions were discussed in a technical paper by E. J. Merrick, a project engineer in the General Electric Company's Missile and Space Vehicle Department at Philadelphia. The paper was presented by Dr. P. R. Knaff, a Department engineering psychologist, at the annual West Coast meeting of the American Astronautical Society.

"The drydocks could be as simple as a plastic 'sausage skin' drawn over and around the craft and then inflated, or they could be as complex as a geodesic sphere," Merrick said.

The drydocks would be inflated with gases fed from the mother ship or from an independent source, providing an atmosphere for sustaining life. When there is work to be performed outside of the spacecraft, the mother ship would be brought into the drydock. Light comfortably suited men could then work inside the sealed "sausage."

After completion of the tasks, the drydock would be pumped out by mechanical collapsing of the structure for return aboard the mother ship. The atmospheric gases of the drydocks would also be returned to the mother ship for purification.

"Safety, reliability, comfort, and efficiency can be increased through the use of such space shelters," Merrick said.

He explained that the "space hog" is a General Electric concept which is designed to provide a comfortable, simple, and highly useful means for excursions outside of the spacecraft. Its name is derived from the cassions used by sandhogs.

The device consists of a pressure-tight, non-rigid tube or cylinder carrying stiffening rings at selected intervals. When not in use, the device is completely stored in an airlock on the spacecraft.

When the device is to be used, it is extended and pressurized. Then the spaceman, wearing a light-duty, comfortable suit, crawls into the space-hog, which can be rotated and/or flexed so that the working end can sweep over any portion of the mother ship to permit close-on, safe task performance. The outboard end of the spacehog would carry such devices as a viewing port, lights, tools, mechanical arms, or an air-lock with traction attachments.

Molecular Biophysics Lectures.—To inquire into the possible physiological role of fast transfer reactions in eliciting specific interactions in ordered macromolecular structures, a series of seminar lectures was held at the Massachusetts Institute of Technology during the spring term of 1960. Included were lectures on electron and energy transfer mechanisms (Kasha, Taube, Murrell, Weber, Stryer) and on current concepts of proton transport mechanisms (Eigen and Grunwald) of immediate biophysical and biochemical significance. Properties of water and ice crucial for proton transfer were reviewed (Frank, Brady, Klotz, Fernandez-Moran). Six lectures of a more general nature dealt with aspects of macromolecular interaction properties (Gergely, Berendsen, Oncley, Davison, Wiener, Elsasser).

Abstracts of these lectures and pertinent references are available gratis, from Professor Francis O. Schmitt, Department of Biology, M.I.T., Cambridge 39, Mass.